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# Off-axis multimode light beam propagation in tapered lenslike media including those with spatial gain or loss variation

Anthony Alan Tovar  
*Portland State University*

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AN ABSTRACT OF THE THESIS OF Anthony Alan Tovar for the Master of Science in Electrical Engineering presented September 29, 1988.

Title: Off-Axis Multimode Light Beam Propagation in Tapered Lenslike Media Including Those with Spatial Gain or Loss Variation.

APPROVED BY THE MEMBERS OF THE THESIS COMMITTEE:

  
Lee W. Casperson, Chair  
Rajinder P. Aggarwal  
Vincent C. Williams

The propagation of light beams in inhomogeneous dielectric media is considered. The derivation begins with first principles and remains general enough to include off-axis asymmetric multimode input beams in tapered lenslike media with spatial variations of gain or loss. The tapering of lenslike media leads to a number of important applications. A parabolic taper is proposed as a model for a heated axially stretched fiber taper, and beams in such media are fully characterized. Other models are proposed by the concatenation of a parabola with other taper functions.

**OFF-AXIS MULTIMODE LIGHT BEAM PROPAGATION IN  
TAPERED LENS LIKE MEDIA INCLUDING THOSE WITH  
SPATIAL GAIN OR LOSS VARIATION**

by  
**ANTHONY TOVAR**

A thesis submitted in partial fulfillment of the  
requirements for the degree of

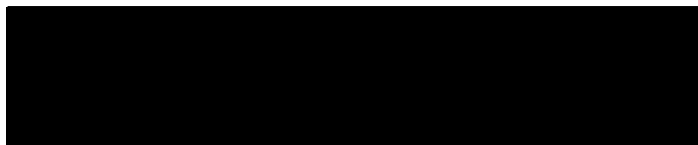
**MASTER OF SCIENCE  
in  
ELECTRICAL ENGINEERING**

**Portland State University**

1988

TO THE OFFICE OF GRADUATE STUDIES:

The members of the Committee approve the thesis of Anthony Alan Tovar presented September 29, 1988.



Lee W. Casperson, Chair



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Lee W. Casperson, Chair, Department of Electrical and Computer Engineering



Bernard Ross, Vice Provost for Graduate Studies

## PREFACE

Fundamental to the philosophy of this work is the idea of simplicity. The thesis starts by stating Maxwell's equations, and concludes with solutions to Maxwell's equations. This implies a natural progression, and the chapters are logically ordered. The first chapter serves as an introduction to the subject of beam propagation in inhomogeneous media. The second and third chapters reduce Maxwell's equations to a set of ordinary differential equations. The second chapter treats fundamental mode propagation while the third chapter considers multimode propagation. The set of ordinary differential equations cannot in general be solved. The fourth chapter overviews methods of finding solutions. The fifth chapter involves interesting ways to combine known solutions.

The mathematics involved generally consists only of simple differentiation. The approximations and constraints made in the derivation are summarized in the summary of the second chapter.

I would like to give very special thanks to my wife, Lyn, for her patience, support, and proofreading of the manuscript. Special thanks must also be given to my advisor, Lee W. Casperson, who taught me the value of simplicity and physical insight. Thanks is also due to the graduate students, faculty, and staff of the electrical engineering department at Portland State University who helped me with things ranging from new ideas to word processing.

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## CHAPTER I

### INTRODUCTION

Since the invention of the laser, there has been renewed interest in the propagation of light in inhomogeneous dielectric media. Research interests in this field have increased even further with the development of relatively inexpensive low-loss glass fibers. Because of the guiding properties of these fibers, optical fiber communication has become a viable alternative to conventional electronic communication.

The geometries of these light guiding systems, or optical waveguides, can often be put into two categories. The thin-film, or slab geometry waveguides, are distinguished by a lack of variation of the index of refraction in one transverse direction. Optical fiber or elliptical optical waveguides are distinguished, in general, by their elliptical cross section. The circular optical waveguide is an important special case because of its circular symmetry. Both the slab and the elliptical geometries have distinct longitudinal and transverse axes as opposed to a spherically inhomogeneous medium such as a Maxwell fish eye or a Luneburg lens [1] which do not have a distinct geometrically imposed longitudinal axis. Not all light guiding geometries fit into either the slab or elliptical categories. An example is a periodic sequence of lenses.

Guiding mechanisms of optical waveguides can also be put into two broad categories. Step-index waveguides are distinguished by a high index of refraction core surrounded by a lower index of refraction cladding with an abrupt change in

the index of refraction. An alternative to this is the graded-index waveguide which has a continuous variation of the index of refraction. An advantage of the step-index waveguide is ease of fabrication. An advantage of the graded-index waveguide is large bandwidth. In addition, graded-index waveguides that have a hyperbolic secant (or in the paraxial approximation, quadratic) variation of the index of refraction have important image-transmitting characteristics [2]-[3], and hence are known as "lenslike" media. The primary focus of this study, as mentioned in the title, will be on such lenslike media with both the slab and elliptical geometries. For brevity, index of refraction shall henceforth be referred to simply as index.

The purpose of this thesis is to study the propagation of light in tapered inhomogeneous dielectric media with as much generality and as few approximations as possible. It will be shown that the analyses here will be general enough to include off-axis input beams, and media with spatial variations of the gain or loss. The approximations made are carefully identified and accompanied with discussions of their validity. To aid successive analysis and synthesis, the studies here will emphasize closed-form analytical results.

Chapter II starts from first principles and reduces the initial partial differential equations to ordinary differential equations for fundamental mode propagation in quadratic-index (lenslike) media, including those with a gain or loss variation. Though this type of analysis has been long known [4], it is important to understand its shortcomings so that progress can be made to improve the theory. With this in mind, this chapter contains a clear enunciation of the approximations, assumptions, and constraints made. To underscore the importance of this philosophy, there is also a discussion of the validity of each of the approximations.

The subject of the third chapter is higher-order mode propagation. Again, the partial differential equation is reduced to a set of ordinary differential equations. The emphasis of the thesis is on tapered lenslike media. For such media, these ordinary differential equations cannot, in general, be solved. However, it will be shown for the first time that, of the nine equations, seven of them can be solved in terms of the remaining two. For media that are tapered in the way along each of the two rectangular transverse axes, the two equations are functionally identical. A special case of this would be a circularly symmetric fiber.

The fourth chapter considers how these two remaining differential equations may be solved for different tapering functions. There are three methods considered. First, it is noticed that the equations exist in other recognizable forms. Second, taper functions are "generated" by assuming solutions. Finally, tables of previously solved equations of the same class are listed. In addition, a new solution is found which corresponds to a physically meaningful taper.

Applications is the subject of the final chapter. The new solution found in the fourth chapter can be concatenated with other optical elements in interesting ways. This leads to a number of important applications.

## **CHAPTER II**

### **FUNDAMENTAL MODE ANALYSIS**

#### **INTRODUCTION**

The purpose of this chapter is to establish the Gaussian beam formalism to be used as foundation for succeeding chapters. As mentioned previously, analytical solutions will be stressed. This chapter focuses on the statements and consequences of the approximations, assumptions, and constraints in the cause of analytic solutions. Ray matrices and Gaussian beam matrices are introduced. Such matrices are often used in paraxial analysis and synthesis of optical systems including resonators, which contain lenses, mirrors, retroreflectors, etc.

#### **MAXWELL'S EQUATIONS**

There are many formalisms used to study the propagation of light, and the first task is to choose one. Among the many choices, four seem most popular: quantum optics [5], the Huygens-Fresnel integral formulation [6], classical electromagnetics [7], and ray optics [8]. Both the Huygens-Fresnel formulation, and ray optics can be viewed as approximations to Maxwell's equations (classical electromagnetics). In the spirit of minimizing approximations, these two formalisms are disregarded. Quantum optics treats the light fields quantum mechanically and involves their quantization. Electromagnetics treats the light fields classically and does not involve their quantization [9]. However, such effects are always small when considering the interaction of light in linear media. On a pragmatic side, the quantized fields are generally more difficult to work with. Hence,

it is conventional to start with Maxwell's equations when solving problems that deal with this subject. With this in mind, Maxwell's equations are a reasonable choice for the studies here.

Even still, it is important to restate that the approximation of classical fields has been made. However, it is also important to realize that Maxwell's equations are derived from purely experimental results. This alone suggests that they are appropriate. However, experimental verification of a specific effect is the ultimate test to any theory. The task now is to state and solve Maxwell's Equations.

According to convention,  $\vec{E}$  represents the vector electric field,  $\vec{D}$  represents the vector electric flux density,  $\vec{H}$  represents the vector magnetic field, and  $\vec{B}$  represents the vector magnetic flux density. In addition,  $\vec{J}$  represents the vector current density and  $\rho$  represents the scalar charge density. With these definitions, Maxwell's Equations in SI units are [7]:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad (2.1)$$

$$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} \quad (2.2)$$

$$\nabla \cdot \vec{D} = \rho \quad (2.3)$$

$$\nabla \cdot \vec{B} = 0 \quad (2.4)$$

These four equations represent the differential form of Ampere's Law, Faraday's Law, Gauss's Law for the electric field, and Gauss's Law for the magnetic field, respectively. For a good introduction to Maxwell's equations and definitions of the field quantities see ref. [10]. Also associated with Maxwell's equations is the continuity equation, which is also known as "conservation of charge":

$$\nabla \cdot \vec{J} = - \frac{\partial \rho}{\partial t} \quad (2.5)$$

It should be stressed that the continuity equation is not independent of Maxwell's

equations. In fact, it is simply derived by taking the divergence of (2.1), realizing that the divergence of the curl of any vector is zero, interchanging order of partial differentiation, and substituting (2.3). The differential form of Maxwell's equations alone is incomplete since it does not contain the boundary condition information that the integral form does. However, these can be derived from the integral form [10]. If  $\bar{i}_n$  designates a unit vector normal to a boundary with no sheet currents or surface charges, then the boundary conditions are:

$$\bar{i}_n \cdot (\bar{D}_1 - \bar{D}_2) = 0 \quad (2.6)$$

$$\bar{i}_n \times (\bar{E}_1 - \bar{E}_2) = 0 \quad (2.7)$$

$$\bar{i}_n \cdot (\bar{B}_1 - \bar{B}_2) = 0 \quad (2.8)$$

$$\bar{i}_n \times (\bar{H}_1 - \bar{H}_2) = 0 \quad (2.9)$$

Now that Maxwell's equations have been stated, something must be said about the material in which the light is to propagate. Such statements are known as "constitutive relations". To be known, all physical properties of any medium must be measured, and hence there always exists experimental error. This error will be manifested in the constitutive relations. With this in mind, it is assumed that the stated constitutive relations approximate a particular medium to within an acceptable tolerance. If this is done, the constitutive relations can often be put in the form:

$$\bar{D} = \epsilon_o \bar{E} + \bar{P} \quad (2.10)$$

$$\bar{J} = \sigma \bar{E} \quad (2.11)$$

$$\bar{B} = \mu_o \bar{H} + \bar{M} \quad (2.12)$$

where  $\epsilon_o$  is the "permittivity of free space" and has the approximate value of  $8.854 \times 10^{-12}$  Farad/meter, and  $\mu_o$  is the "permeability of free space" which has the exact value  $4\pi \times 10^{-7}$  Henry/meter. The symbols  $\bar{P}$  and  $\bar{M}$  represent the Polarization and Magnetization of the medium, respectively. Equation (2.11) is known as the

microscopic Ohm's law. In general  $\bar{P}$  and  $\bar{M}$  are vectors, and  $\mu$  and  $\sigma$  are tensors which are all functions of space, time, and any of the field quantity magnitudes. If the medium is magnetically isotropic, then the magnetization will be parallel to the vector magnetic field:

$$\bar{M} = \mu_o \chi_m \bar{H} \quad (2.13)$$

If the fields are monochromatic, they can be written in the form

$$\bar{E} \equiv \text{Re} \{ \bar{E}'(x, y, z) e^{i\omega t} \} \quad (2.14)$$

$$\bar{H} \equiv \text{Re} \{ \bar{H}'(x, y, z) e^{i\omega t} \} \quad (2.15)$$

where  $\text{Re}\{\}$  designates the real part of the complex bracketed quantity, and the primes indicate complex time-independent fields. Furthermore, the medium will have gain or loss if there is an out of phase component of the polarization [9]:

$$\bar{P} = \bar{C} \cos(\omega t) + \bar{S} \sin(\omega t) \quad (2.16)$$

If  $\text{Re}\{\}$  designates the real part of the complex bracketed quantity, then

$$\bar{P} = \text{Re} \{ \bar{C} e^{i\omega t} - i \bar{S} e^{i\omega t} \} \quad (2.17)$$

$$= \text{Re} \{ (\bar{C} - i \bar{S}) e^{i\omega t} \} \quad (2.18)$$

If the material is electrically isotropic then

$$\bar{P} = \text{Re} \{ \epsilon_o \chi_e \bar{E}'(x, y, z) e^{i\omega t} \} \quad (2.19)$$

Depending on the application, the assumption that the medium is electrically isotropic may be difficult to justify. In the case of optical waveguides, anisotropy may be induced by external stresses, be they intentional or not, or by internal stresses caused during manufacturing. There has been an analysis of anisotropic media where transverse spatial index variations have been included as a perturbation [11].

It is important to understand the ramifications of assuming monochromatic fields. This analysis does not apply to waveguides illuminated by nonmonochromatic sources such as light bulbs. Unfortunately, neither does this apply



exactly to lasers since they contain spontaneous emission noise which make them nonmonochromatic. However, in practice, the spontaneous emission laser output is much smaller than the stimulated emission output, and therefore, to a good approximation, lasers are monochromatic. Care must still be taken for some lasers, like semiconductor lasers, which have a larger spontaneous emission output. Fields that are monochromatic are varying in time harmonically. Mode-locked lasers and lasers operated in an instability regime yield output which cannot be called harmonic.

Definitions (2.14) and (2.15) and constraints (2.13) and (2.19) can be inserted into the constitutive relations (2.10) to (2.12), and Maxwell's equations (2.1) through (2.4). The real parts of the complex equations are satisfied, but no information is gained about the imaginary parts. Without loss of generality, the imaginary parts of these equations may also be equated. In particular, constitutive relations (2.10) to (2.12) can be rewritten

$$\bar{D}' = \epsilon \bar{E}' \quad (2.20)$$

$$\bar{J}' = 0 \quad (2.21)$$

$$\bar{B}' = \mu \bar{H}' \quad (2.22)$$

where  $\epsilon \equiv \epsilon_o + \epsilon_o \chi_e$  and  $\mu \equiv \mu_o + \mu_o \chi_m$ . It has been assumed that the medium is nonconductive (i.e.  $\sigma = 0$ ). Therefore any losses that exist shall be constrained to be manifested in the imaginary part of the permittivity as "negative gain". One type of loss is absorption. Unfortunately, in general, absorption is complicated. Firstly, attenuation due to absorption is wavelength dependent. Worse yet is the fact that, because of lack of information, this dependence cannot be derived from first principles. Secondly, any absorption will heat up the medium increasing the amount of localized temperature inhomogeneities, which will change the index of refraction of the medium. Another loss mechanism is scattering. Scattering is caused

by random spatial fluctuations in the density of the medium. The typical scattering mechanism in an optical waveguide is Rayleigh scattering. Attenuation due to such scattering is proportional to  $\lambda^{-4}$ . Another concern is coupling. If the input beam is launched from freespace into an inhomogeneous medium, some of the energy of the beam will be reflected. Similarly, dispersion is of concern [12]. It will further be assumed that there exists no net unbalanced charge, and hence the charge density is identically zero (i.e.  $\rho = 0$ ). Similarly, it will be assumed that there exist no currents and hence  $\bar{J} = 0$ . Note that the continuity equation is then always satisfied since both sides of (2.5) are zero. Under these conditions, the medium is said to be source-free.

With these definitions and assumptions, Maxwell's equations can be rewritten in this "monochromatic, source-free, isotropic" form:

$$\nabla \times \bar{H}' = i\omega\epsilon\bar{E}' \quad (2.23)$$

$$\nabla \times \bar{E}' = -i\omega\mu\bar{H}' \quad (2.24)$$

$$\nabla \cdot \bar{E}' = -\frac{\nabla\epsilon \cdot \bar{E}'}{\epsilon} \quad (2.25)$$

$$\nabla \cdot \bar{H}' = -\frac{\nabla\mu \cdot \bar{H}'}{\mu} \quad (2.26)$$

It should be pointed out that the divergence of each side of (2.24) yields (2.26) and the divergence of each side of (2.23) gives (2.25). It should also be noted that these equations are valid for nonlinear media in the sense that  $\mu$  and  $\epsilon$  may vary with the other time-independent field quantities.

To obtain the power flowing through a surface  $A$  with an outward-directed unit normal vector  $\bar{i}_n$  at each point, the surface integral

$$P = \int_A \bar{S} \cdot \bar{i}_n dA \quad (2.27)$$

must be evaluated where

$$\bar{S} \equiv \bar{E} \times \bar{H} \quad (2.28)$$

However, one is often interested in the time-average power, which for sinusoidally time-varying fields is [13]

$$\langle \bar{S} \rangle = \frac{1}{2} \text{Re } \bar{E}' \times \bar{H}'^* \quad (2.29)$$

where the asterisk represents complex conjugate.

## THE SCALAR WAVE EQUATION

Materials chosen for their dielectric properties are usually magnetically simple. Hence, for many media, it is reasonable to assume magnetic linearity and homogeneity (i.e.  $\mu$  is space independent). Taking the curl of Faraday's Law yields:

$$\begin{aligned} \nabla \times \nabla \times \bar{E}' &= \nabla \times (-i\omega\mu\bar{H}') \\ &= -i\omega\mu (\nabla \times \bar{H}') \\ &= -i\omega\mu (i\omega\epsilon\bar{E}') \\ &= \omega^2\mu\epsilon\bar{E}' \end{aligned} \quad (2.30)$$

But by definition of the Laplacian of a vector,

$$\nabla \times \nabla \times \bar{E} \equiv \nabla(\nabla \cdot \bar{E}) - \nabla^2 \bar{E} \quad (2.31)$$

Therefore,

$$\nabla^2 \bar{E}' + \omega^2\mu\epsilon\bar{E}' - \nabla(\nabla \cdot \bar{E}') = 0 \quad (2.32)$$

One name often associated with (2.32) is the "vector wave equation". It should be noted that, strictly speaking, (2.32) is not a wave equation at all since wave equations are by definition time and space-dependent differential equations.  $\bar{E}'$  describes a wave only after it is multiplied by  $e^{i\omega t}$  and the real part is taken. Therefore, another common name for (2.32) is the "vector Helmholtz equation" in honor of the German physician, philosopher, mathematician, and physicist Hermann Ludwig Ferdinand von Helmholtz. For simplicity and convention, however,

the time-independent differential equations here and to follow shall still be referred to as wave equations. It is the author's preference to state the vector wave equation in terms of index of refraction.

Plane wave index of refraction can be defined

$$n \equiv \frac{c}{v} = \frac{c}{1/\sqrt{\mu\epsilon}} \quad (2.33)$$

$$\epsilon = \frac{n^2}{\mu c^2} \quad (2.34)$$

$$\nabla\epsilon = \frac{2}{\mu c^2} n \nabla n \quad (2.35)$$

The symbol  $c$  representing the speed of light in vacuum, comes from the Latin word *celer*, meaning fast, and has the exact value of 2.99792458 meters/second since the meter was redefined in 1983 [6]. It should be noted that since  $c \equiv 1/\sqrt{\mu_0\epsilon_0}$  and  $\mu_0$  has an exact value, there exists a defined exact value for  $\epsilon_0$  as well. Inserting equations (2.33) through (2.35) into (2.25) yields Gauss's law for the electric field in terms of index of refraction:

$$\nabla \cdot \vec{E}' = -2 \frac{\nabla n}{n} \cdot \vec{E}' \quad (2.36)$$

Taking the gradient of both side yields:

$$\nabla (\nabla \cdot \vec{E}') = -2 \frac{n \nabla(\nabla n \cdot \vec{E}') - (\nabla n \cdot \vec{E}') \nabla n}{n^2} \quad (2.37)$$

Combining this with (2.32) and the definitions of index of refraction (2.33) gives:

$$\nabla^2 \vec{E}' + \left(\frac{\omega^2}{c^2} n^2\right) \vec{E}' + 2 \frac{\nabla(\nabla n \cdot \vec{E}')}{n} - 2 \frac{\nabla n}{n^2} (\nabla n \cdot \vec{E}') = 0 \quad (2.38)$$

It is common to introduce the plane wave number,  $k$ , which is constant here.

$$k \equiv \frac{\omega}{c} \quad (2.39)$$

Therefore,

$$\nabla^2 \vec{E}' + (k^2 n^2) \vec{E}' + 2 \frac{\nabla(\nabla n \cdot \vec{E}')}{n} - 2 \frac{\nabla n}{n^2} (\nabla n \cdot \vec{E}') = 0 \quad (2.40)$$

This is a second order, linear, homogeneous, non-constant coefficient, vector, partial differential equation with three independent variables. From here on this equation shall be referred to as the vector wave equation.

Before proceeding, a coordinate system must be chosen for both the electric field vector, and for each component of the electric field. Since the Laplacian of a vector is simplest in rectangular coordinates, the vector electric field shall be stated in these terms. Again, for simplicity, the index of refraction shall be stated in this same coordinate system. Suppose then that

$$\vec{E}' = E_x \vec{i}_x + E_y \vec{i}_y + E_z \vec{i}_z \quad (2.41)$$

Consider the alphath rectangular component of the electric field vector:

$$\begin{aligned} \nabla^2 E_\alpha + (k^2 n^2) E_\alpha - \frac{2}{n^2} \frac{\partial n}{\partial \alpha} \left( \frac{\partial n}{\partial x} E_x + \frac{\partial n}{\partial y} E_y + \frac{\partial n}{\partial z} E_z \right) \\ + \frac{2}{n} \frac{\partial}{\partial \alpha} \left( \frac{\partial n}{\partial x} E_x + \frac{\partial n}{\partial y} E_y + \frac{\partial n}{\partial z} E_z \right) = 0 \end{aligned} \quad (2.42)$$

This illustrates the fact that the vector wave equation can also be interpreted as three scalar equations. The difficulty encountered is that they are undeniably coupled. Typically, one decouples the equations, and then solves them one at a time. Unfortunately, an approximation is made in the decoupling process. If

$$\frac{2}{n^2} \left[ \frac{\partial n}{\partial \alpha} \right]^2 \ll k^2 n^2 \quad (2.43)$$

then

$$\nabla^2 E_\alpha + (k^2 n^2) E_\alpha \approx 0 \quad (2.44)$$

This is the scalar wave equation and it is the principal result of this section. Often this is considered the starting point for many derivations in the field of beam optics. To better understand the approximation (2.43), note that to first order

$$k \sim \frac{1}{\lambda} \quad n \sim 1 \quad (2.45)$$

Using this, and averaging the derivative from zero to the core radius in the  $\alpha$  direction, (2.43) can be rewritten as

$$\frac{\Delta n}{\Delta \alpha} \ll \frac{1}{\lambda} \quad (2.46)$$

which is read: the index of refraction varies negligibly in the distance of a wavelength. However, one must be careful when considering the polarization. If the index of refraction varies at all, the polarization will "drift". It is clear from the coupling demonstrated in (2.42) that the polarization will, in general, change with the propagation of the beam. However, (2.44) states that in rectangular coordinates, the polarization will not change. Therefore, it can be concluded that some important polarization information has been lost. Note that this approximation is not unlike a geometric optics approximation. If the wavelength approaches zero, then there cannot be any index variations in the distance of a wavelength. For a good discussion of this approximation, see [13],[14].

There do exist waveguides which do not change the polarization of the beam. They are known as "polarization preserving" fibers [15]. However, these fibers are made by inducing a large amount of anisotropy. This study is not meant to be exhaustive, and anisotropy is one of the many issues not considered here.

## THE PARAXIAL WAVE EQUATION

Suppose that the beam is linearly polarized in the  $x$  direction, then

$$\nabla^2 E_x + (k^2 n^2) E_x = 0 \quad (2.47)$$

If  $T$  represents the two transverse coordinates, then  $n$  can always be put in the form

$$n \equiv n_o(z) - \frac{1}{2}n'(T,z) \quad (2.48)$$

which implies that

$$n^2 = [n_o(z) - \frac{1}{2}n'(T,z)]^2 \quad (2.49)$$

Typically the variations of index of refraction are small, and a first order binomial expansion is justified:

$$n^2 \approx n_o^2 - n_o n' \quad (2.50)$$

This approximation is known as the "weak media" approximation. Now, (2.47) can be written

$$\nabla^2 E_x + (k^2 n_o^2 - k^2 n_o n') E_x = 0 \quad (2.51)$$

The form of the equation can be changed by a substitution.

$$E_x \equiv A(T, z) e^{-ik \int n_o dz} \quad (2.52)$$

Calculating the necessary derivative:

$$\frac{\partial^2 E_x}{\partial z^2} = \left[ \frac{\partial^2 A}{\partial z^2} - 2ikn_o \frac{\partial A}{\partial z} - ik \frac{\partial n_o}{\partial z} A - k^2 n_o^2 A \right] e^{-ik \int n_o dz} \quad (2.53)$$

which yields

$$\nabla_T^2 A + \frac{\partial^2 A}{\partial z^2} - 2ikn_o \frac{\partial A}{\partial z} - (ik \frac{\partial n_o}{\partial z} + k^2 n_o n') A = 0 \quad (2.54)$$

where  $\nabla_T^2 A \equiv \nabla^2 A - \frac{\partial^2 A}{\partial z^2}$ . For nearly plane waves,

$$\left| \frac{\partial^2 A}{\partial z^2} \right| \ll \left| 2ikn_o \frac{\partial A}{\partial z} \right| \quad (2.55)$$

which implies that

$$\nabla_T^2 A - 2ikn_o \frac{\partial A}{\partial z} - (ik \frac{\partial n_o}{\partial z} + k^2 n_o n') A = 0 \quad (2.56)$$

This approximation is known as the paraxial approximation, and this equation is known as the paraxial wave equation. In a homogeneous medium,  $n'$  is zero, and  $n_o$  is a complex constant. Under these conditions a solution to (2.56) is a complex constant, and the electric field can be obtained from (2.52). Because we have assumed the medium is "weak", it is logical to assume the homogeneous medium solution multiplied by a slowly varying envelope. Therefore, the weak medium approximation and the paraxial approximation go hand in hand. The phase fronts

of the homogeneous medium solution are flat (infinite in radius of curvature), hence they are known as "plane waves". Therefore, this approximation is sometimes known as the "nearly plane wave" approximation.

In tapered media, one expects that some of the energy of the input beam will be reflected both back toward the input, and out the "sides" of the waveguide. In fact even the ray theory will predict this. This can be seen by considering an off-axis ray in a strongly tapered, cladded waveguide represented schematically in figure 2-1. However, the weak medium assumption does not allow for such reflections. Therefore such reflections are assumed to be negligibly small. This can only be assumed if the tapering is "weak".

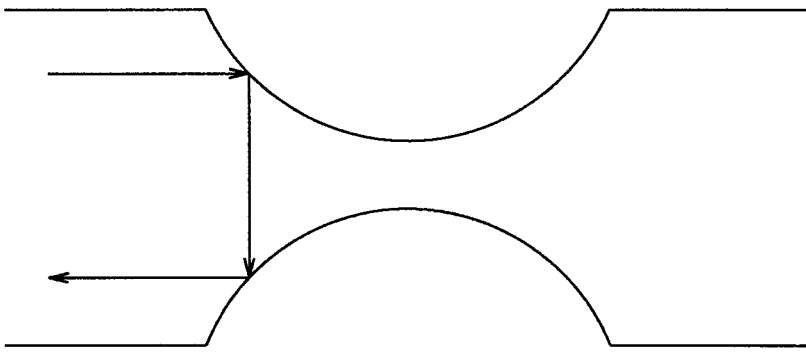


Figure 2-1. Ray Reflection in a Strongly Tapered Cladded Medium.

## FUNDAMENTAL MODE SOLUTIONS

It will be shown that the paraxial wave equation can be reduced to a set of ordinary differential equations without any further approximations if

$$n' = n_{1x}(z)x + n_{1y}(z)y + n_{xy}(z)xy + n_{2x}(z)x^2 + n_{2y}(z)y^2 \quad (2.57)$$

This is the most general quadratic profile possible. This profile has several advantages. In the paraxial approximation, ideal images are formed, and hence there is



great interest in this profile from the optical engineering community. In fact there have been several issues of the journal 'Applied Optics' dedicated to these "GRIN" (GRaded-INdex) materials [16]. This is why such materials are sometimes referred to as "lenslike". Perhaps its most promising feature is its generality. To within some approximation, all functions are quadratic, at least locally. This property has been used to analyze gas lasers. Gas lasers often have spatial temperature inhomogeneities. However, as mentioned previously, this gives rise to a radial index profile. It has been shown that the index has a weak zero-order Bessel function profile [17]. For purposes of analysis this function can be Taylor series expanded. Near the center, cubic and higher order terms can be shown to be negligible. Hence, the spatial behavior of many high power gas lasers may be approximated by a quadratic-index profile [18]. As previously suggested, a hyperbolic secant index profile can be shown to produce ideal images [12]. It has been shown that such waveguides are manufacturable [19]. Of course, a quadratic is a reasonable first-order approximation to a hyperbolic secant.

If the longitudinal axis is chosen to be the centroid of the cross-section of the waveguide, then  $n_{1x}(z)$  and  $n_{1y}(z)$  allow the index variation to be centered off of the waveguide axis in the  $x$  and  $y$  directions respectively. A nonconstant  $n_{xy}(z)$  allows the axis of the index of refraction to twist. If  $n_{2y}(z)$  is not equal to  $n_{2x}(z)$ , and both are nonzero, then the geometry is elliptical. If  $n_{2y}(z)$  is equal to  $n_{2x}(z)$ , and nonzero, then the geometry is said to be circular. In the special case of the slab geometry,  $n_{1y}(z)$ ,  $n_{xy}(z)$ , and  $n_{2y}(z)$  are all zero. It should be noted that the continuous quadratic index variation implied by (2.57) can only be valid inside some finite radius. Outside this finite radius, Maxwell's equations must be solved again, and the solutions must match at this "boundary" as indicated by equations (2.6) to (2.9). However, if this is done, one finds that the solutions are very similar to the solutions gained

without matching boundary conditions. That is, for purposes of calculating the magnitudes of the fields, one need not match boundary conditions. The solutions obtained without matching boundary conditions are known as the infinite Gaussian solutions. Those obtained that include matched boundary conditions are referred to as truncated Gaussians. One problem that exists with the infinite Gaussian solutions is that they cannot predict bending losses.

The  $z$ -dependence of the coefficients in (2.57) allow for longitudinal index variation. However, it should be pointed out that these coefficients may also be temperature and wavelength dependent. In some optical waveguides, this temperature dependence is sensitive enough to cause noticeable index changes due simply to human contact.

The paraxial wave equation with index profile (2.57) is

$$\nabla_T^2 A - 2ikn_o \frac{\partial A}{\partial z} - [ik \frac{\partial n_o}{\partial z} + k^2 n_o (n_{1x}x + n_{1y}y + n_{xy}xy + n_{2x}x^2 + n_{2y}y^2)]A = 0 \quad (2.58)$$

Time should be taken to examine this equation more closely. First, consider the types of solutions allowed by (2.58). The most obvious one is the trivial solution. It is important to note that it must exist. In the case of noncomplex index materials, all electromagnetic sources have been eliminated from the problem. This is one of the results of neglecting the charge density,  $\rho$ . Hence, if the boundary conditions demand zero fields at the input of the medium, the output must also be identically zero. Note that the charge density makes Gauss's law for the electric field a forced differential equation, and that (2.58) is not forced. That is, as mentioned above, there are no sources.

One may consider the one-dimensional (1D) analog of (2.58):

$$A'' + (\alpha_1 + \alpha_2 x + \alpha_3 x^2)A = 0 \quad (2.59)$$

But this is the same equation as the 1D quantum mechanical harmonic oscillator which has Hermite-Gaussians as solutions [9]. By analogy, a Gaussian substitution may be useful:

$$A(x,y,z) = B(x,y,z) \exp -i \left[ \frac{Q_x(z)}{2} x^2 + \frac{Q_y(z)}{2} y^2 + Q_{xy}(z)xy + S_x(z)x + S_y(z)y + P(z) \right] \quad (2.60)$$

which implies that

$$\begin{aligned} & \left[ \frac{\partial^2 B}{\partial x^2} - 2i (Q_x x + Q_{xy} y + S_x) \frac{\partial B}{\partial x} \right] + \left[ \frac{\partial^2 B}{\partial y^2} - 2i (Q_y y + Q_{xy} x + S_y) \frac{\partial B}{\partial y} \right] - 2ikn_o \frac{\partial B}{\partial z} \\ & - \left\{ [Q_x^2 + Q_{xy}^2 + kn_o \frac{\partial Q_x}{\partial z} + k^2 n_o n_{2x}] x^2 + [Q_y^2 + Q_{xy}^2 + kn_o \frac{\partial Q_y}{\partial z} + k^2 n_o n_{2y}] y^2 \right. \\ & + [Q_{xy} (Q_x + Q_y) + kn_o \frac{\partial Q_{xy}}{\partial z} + \frac{1}{2} k^2 n_o n_{xy}] 2xy + [Q_x S_x + Q_{xy} S_y + kn_o \frac{\partial S_x}{\partial z} + \frac{1}{2} k^2 n_o n_{1x}] 2x \\ & \left. + [Q_y S_y + Q_{xy} S_x + kn_o \frac{\partial S_y}{\partial z} + \frac{1}{2} k^2 n_o n_{1y}] 2y + [2kn_o \frac{\partial P}{\partial z} + S_x^2 + S_y^2 + i (Q_x + Q_y) + ik \frac{\partial n_o}{\partial z}] \right\} B = 0 \end{aligned} \quad (2.61)$$

Note that if the coefficient linear in  $B$  is zero, then one of the solutions to the differential equation is a constant. This solution corresponds to the fundamental mode of the waveguide. Setting the coefficient to the  $B$  term to zero implies that

$$[Q_x^2 + Q_{xy}^2 + kn_o \frac{\partial Q_x}{\partial z} + k^2 n_o n_{2x}] = 0 \quad (2.62)$$

$$[Q_y^2 + Q_{xy}^2 + kn_o \frac{\partial Q_y}{\partial z} + k^2 n_o n_{2y}] = 0 \quad (2.63)$$

$$[Q_{xy} (Q_x + Q_y) + kn_o \frac{\partial Q_{xy}}{\partial z} + \frac{1}{2} k^2 n_o n_{xy}] = 0 \quad (2.64)$$

$$[Q_x S_x + Q_{xy} S_y + kn_o \frac{\partial S_x}{\partial z} + \frac{1}{2} k^2 n_o n_{1x}] = 0 \quad (2.65)$$

$$[Q_y S_y + Q_{xy} S_x + kn_o \frac{\partial S_y}{\partial z} + \frac{1}{2} k^2 n_o n_{1y}] = 0 \quad (2.66)$$

$$[2kn_o \frac{\partial P}{\partial z} + S_x^2 + S_y^2 + i (Q_x + Q_y) + ik \frac{\partial n_o}{\partial z}] = 0 \quad (2.67)$$

Equations (2.62) and (2.63) are known as beam parameter equations, and  $Q_x$  and  $Q_y$  are referred to as beam parameters. The function  $P(z)$  is known as the phase parameter and equation (2.67) is known as the phase parameter equation. The

beam parameter is often divided by  $kn_o$ , and broken into real and imaginary parts:

$$\frac{1}{q_x} \equiv \frac{Q_x}{kn_o} = \frac{1}{R_x} - i \frac{2}{kn_o w_x^2} \quad (2.68)$$

The symbol  $q_x$  is known as the complex beam radius (note that it has units of distance). It can be shown that  $R_x$  represents the radius of curvature of the phase fronts of the beam [20] which are defined to be positive when the beam is diffracting. Similarly,  $w_x$  is the distance at which the electric field amplitude drops to  $1/e$  of its maximum value in the  $x$  direction. The quantity  $w_x$  has the physically meaningful name "spotsize". Similar results hold for  $Q_y$ . In a homogeneous medium  $w_x$  and  $R_x$  are governed by particularly simple formulas if the place where  $w_x$  is minimum (called the "waist") is at the origin [20]:

$$w_x^2(z) = w_{ox}^2 \left[ 1 + \left( \frac{\lambda_m z}{\pi w_{ox}^2} \right)^2 \right] = w_{ox}^2 \left[ 1 + \left( \frac{z}{z_o} \right)^2 \right] \quad (2.69)$$

$$R_x^2(z) = z^2 \left[ 1 + \left( \frac{\pi w_{ox}^2}{\lambda_m z} \right)^2 \right] = z^2 \left[ 1 + \left( \frac{z_o}{z} \right)^2 \right] \quad (2.70)$$

The quantity  $z_o$  is called the "Rayleigh length" and is defined as  $\pi w_o^2 / \lambda_m$ . The divergence angle of the beam can be found by letting  $z$  be much greater than this Rayleigh length:

$$w(z \gg z_o) = \lambda_m z / \pi w_o \quad (2.71)$$

$$\theta \equiv 2 \frac{dw}{dz} = \frac{2\lambda_m}{\pi w_o^2} \quad (2.72)$$

Hence, the Rayleigh length is often interpreted physically as the length at which diffraction is not negligible. As a side note, it can be shown that, for real media,  $w_x$  will be minimum when  $R_x$  is infinite.

The remaining equation for  $B(x, y, z)$  is

$$\left[ \frac{\partial^2 B}{\partial x^2} - 2i (Q_x x + Q_{xy} y + S_x) \frac{\partial B}{\partial x} \right] + \left[ \frac{\partial^2 B}{\partial y^2} - 2i (Q_y y + Q_{xy} x + S_y) \frac{\partial B}{\partial y} \right] - 2ikn_o \frac{\partial B}{\partial z} = 0 \quad (2.73)$$

A nontrivial solution to this equation is, as mentioned above, a complex constant. If this constant is written in polar form, then

$$\bar{E}' = \{E_o e^{-i\phi_o} e^{-ik \int n_o dz} e^{-i[1/2 Q_x(z)x^2 + 1/2 Q_y(z)y^2 + Q_{xy}(z)xy + S_x(z)x + S_y(z)y + P(z)]}\} \bar{i}_x \quad (2.74)$$

and

$$\bar{E} = \text{Re} \{E_o e^{-i\phi_o} e^{-ik \int n_o dz} e^{-i[1/2 Q_x(z)x^2 + 1/2 Q_y(z)y^2 + Q_{xy}(z)xy + S_x(z)x + S_y(z)y + P(z)]} e^{i\omega t}\} \bar{i}_x \quad (2.75)$$

The corresponding magnetic fields can be solved via (2.24) and (2.22).

The Gaussian part of the solution can be written as

$$e^{-i[1/2 Q_{xx}(x-d_{xp})^2 + 1/2 Q_{yy}(y-d_{yp})^2 - 1/2 Q_{xx}d_{xp}^2 - 1/2 Q_{yy}d_{yp}^2 + P_r]} e^{[1/2 Q_{xx}(x-d_{xa})^2 + 1/2 Q_{yy}(y-d_{ya})^2 - 1/2 Q_{xx}d_{xa}^2 - 1/2 Q_{yy}d_{ya}^2 + P_i]}$$

where

$$d_{xa} = -S_{xi}/Q_{xi} \quad (2.76)$$

$$d_{xp} = -S_{xr}/Q_{xr} \quad (2.77)$$

and the  $r$  and  $i$  subscripts denote real and imaginary parts respectively. It is not difficult to see that  $d_{xa}$  represents the displacement of the amplitude center of the beam in the  $x$ -direction. In the same way,  $d_{xp}$  represents the displacement of the phase center of the beam in the  $x$ -direction. Similar results hold for  $d_{ya}$  and  $d_{yp}$ . It is clear from (2.75) that the field quantities must be real. Therefore, (2.75) can be rewritten

$$\bar{E} = E_o e^{k \int n_o dz + [1/2 Q_{xx}(x-d_{xa})^2 + 1/2 Q_{yy}(y-d_{ya})^2 - 1/2 Q_{xx}d_{xa}^2 - 1/2 Q_{yy}d_{ya}^2 + P_i]} \times \cos \left[ 1/2 Q_{xx}(x-d_{xp})^2 + 1/2 Q_{yy}(y-d_{yp})^2 - 1/2 Q_{xx}d_{xp}^2 - 1/2 Q_{yy}d_{yp}^2 + P_r + k \int n_o dz - \omega t + \phi_o \right] \bar{i}_x \quad (2.78)$$

Even for  $n_o, n_{1x}, n_{1y}, n_{xy}, n_{2x}, n_{2y}$  constant, equations (2.62) - (2.67) are difficult to solve. However, for elliptic beams whose major axis is parallel or perpendicular to the major axis of the index variation in the medium,  $Q_{xy}(0) = 0$ . If  $n_{xy} = 0$ , the trivial solution is the solution to (2.64). The resulting set of equations is

$$[Q_x^2 + kn_o \frac{\partial Q_x}{\partial z} + k^2 n_o n_{2x}] = 0 \quad (2.79)$$

$$[Q_y^2 + kn_o \frac{\partial Q_y}{\partial z} + k^2 n_o n_{2y}] = 0 \quad (2.80)$$

$$[Q_x S_x + kn_o \frac{\partial S_x}{\partial z} + \frac{1}{2} k^2 n_o n_{1x}] = 0 \quad (2.81)$$

$$[Q_y S_y + kn_o \frac{\partial S_y}{\partial z} + \frac{1}{2} k^2 n_o n_{1y}] = 0 \quad (2.82)$$

$$[2kn_o \frac{\partial P}{\partial z} + S_x^2 + S_y^2 + i(Q_x + Q_y) + ik \frac{\partial n_o}{\partial z}] = 0 \quad (2.83)$$

Interestingly, the phase parameter equation (2.67) remains unchanged. Note that the equations are decoupled in the sense that one need only solve one at a time. The inciteful substitution

$$Q_\alpha = kn_o \frac{1}{r_\alpha} \frac{\partial r_\alpha}{\partial z} \quad (2.84)$$

puts the equations in a form not difficult to solve for  $z$ -independent index. If the letter  $\alpha$  represents either  $x$  or  $y$ , and  $\gamma = (n_{2\alpha}/n_o)^{1/2}$ , then

$$\frac{Q_\alpha}{kn_o} = \frac{-\gamma \sin \gamma z + [Q_\alpha(0)/kn_o] \cos \gamma z}{\cos \gamma z + \gamma^{-1} [Q_\alpha(0)/kn_o] \sin \gamma z} \quad (2.85)$$

Furthermore, if

$$S'_\alpha = S_\alpha - \frac{n_{1\alpha} Q_\alpha}{2n_{2\alpha}} \quad (2.86)$$

and

$$P' = P - \frac{n_{1x} S_x}{2n_{2x}} - \frac{n_{1y} S_y}{2n_{2y}} + \frac{n_{1x}^2 Q_x}{8n_{2x}^2} + \frac{n_{1y}^2 Q_y}{8n_{2y}^2} \quad (2.87)$$

then [21]  $S'_\alpha$  and  $P'$  have the solutions

$$S'_x = \frac{S'_x(0)}{\cos \gamma_x z + (Q_x(0)/kn_o)/\gamma_x \sin \gamma_x z} \quad (2.88)$$

$$\begin{aligned} P' = & P'(0) - i/2 \ln \{ \cos \gamma_x z + (Q_x(0)/kn_o)/\gamma_x \sin \gamma_x z \} \\ & - i/2 \ln \{ \cos \gamma_y z + (Q_y(0)/kn_o)/\gamma_y \sin \gamma_y z \} \\ & - \frac{[S'_x(0)]^2}{2kn_o} \frac{(\sin \gamma_x z)/\gamma_x}{\cos \gamma_x z + (Q_x(0)/kn_o)/\gamma_x \sin \gamma_x z} \\ & - \frac{[S'_y(0)]^2}{2kn_o} \frac{(\sin \gamma_y z)/\gamma_y}{\cos \gamma_y z + (Q_y(0)/kn_o)/\gamma_y \sin \gamma_y z} \end{aligned} \quad (2.89)$$

Such a beam is known as a Gaussian beam because of its transverse spatial

distribution. The Gaussian function arises from the paraxial wave equation (2.56) and the parabolic index profile (2.57). For index profiles that are not quadratic or special cases of a quadratic, the resulting beams will not be Gaussian. It has been shown that for nonparabolic index profiles, the Gaussian beam is a good approximation for the fundamental mode [22].

Intuition suggests that the displacement of the beam should travel in the same way a ray would, and hence  $d_{xa}$  and  $d_{ya}$  should satisfy the paraxial ray equation. It has been shown rigorously that if the index of refraction is real, then  $d_{xa}$  and  $d_{ya}$  are indeed governed by the paraxial ray equation for "weak" lenslike media [21]:

$$\frac{d}{dz} \left[ n \frac{d\vec{r}}{dz} \right] = \nabla_T n \quad (2.90)$$

For paraxial ray systems, position and slope at any point in a given medium can be found in terms of the position and slope at another point. Therefore, there exists a useful matrix formalism:

$$\begin{bmatrix} r_2 \\ r'_2 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} r_1 \\ r'_1 \end{bmatrix} \quad (2.91)$$

where  $r_2$  denotes ray position and  $r'_2$  represents ray slope. This 2x2 matrix has been calculated for several optical elements (see Table 2.1). In the thin lens matrix, the focal length,  $f$ , of the thin lens is defined to be positive when the lens is converging.

TABLE I  
ABCD MATRICES FOR COMMON OPTICAL ELEMENTS

medium or boundary	ABCD matrix
homogeneous medium (length $d$ )	$\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$
thin lens	$\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$
curved dielectric boundary	$\begin{bmatrix} 1 & 0 \\ \frac{n_2 - n_1}{n_2} \frac{1}{R} & n_1/n_2 \end{bmatrix}$
spherical mirror	$\begin{bmatrix} 1 & 0 \\ -2/R & 1 \end{bmatrix}$
cornercube	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
Untapered complex lenslike media $\gamma = (k_2/k_0)^{1/2}$	$\begin{bmatrix} \cos(\gamma z) & \gamma^{-1} \sin(\gamma z) \\ -\gamma \sin(\gamma z) & \cos(\gamma z) \end{bmatrix}$
apodized aperture	$\begin{bmatrix} 1 & 0 \\ -i\lambda/\pi w_a^2 & 1 \end{bmatrix}$
Tapered complex lenslike media $W = v(z_1)u'(z_1) - u(z_1)v'(z_1)$	$\frac{1}{W} \begin{bmatrix} u'(z_1)v(z) - v'(z_1)u(z) & v(z_1)u(z) - u(z_1)v(z) \\ u'(z_1)v'(z) - v'(z_1)u'(z) & v(z_1)u'(z) - u(z_1)v'(z) \end{bmatrix}$
Tapered complex lenslike media (WKB Approximation) $\gamma(z) = \int_0^z (k_2(z')/k_0)^{1/2} dz'$	$\begin{bmatrix} (k_2(z)/k_2(0))^{-1/4} \cos \gamma(z) & (k_2(z)k_2(0)/k_0^2)^{-1/4} \sin \gamma(z) \\ -(k_2(z)k_2(0)/k_0^2)^{1/4} \sin \gamma(z) & (k_2(z)/k_2(0))^{1/4} \cos \gamma(z) \end{bmatrix}$



For the tapered complex lenslike medium matrix,  $u(z)$  and  $v(z)$  correspond to the two linearly independent solutions of

$$r'' + k_2(z)/k_o r = 0 \quad (2.92)$$

It has been shown [23] that the ABCD matrix that corresponds to the solution of (2.92), regardless of the variation of  $k_2(z)$  is unimodular (i.e. unit determinant). The WKB approximation assumes that the nonconstant coefficient varies negligibly in a wavelength of the oscillatory solution [24]. It can also be shown for beams that

$$\frac{1}{q_{2x}} = \frac{C + D/q_{1x}}{A + B/q_{1x}} \quad (2.93)$$

where  $1/q_{2x} \equiv Q_x/k_o = 1/R - i\lambda_m/\pi w^2$ , and A, B, C, and D are the same elements as the ray matrices above! There is no obvious reason to expect that the ray theory and the beam theory should be connected in this way. However, they are. This result (2.93) is known as the Kogelnik transformation (or more commonly, the "ABCD law").

The real effectiveness of the matrix theory lies in its ability to combine elements. For ray matrices it is easy to consider:

$$\begin{bmatrix} r_3 \\ r'_3 \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} r_2 \\ r'_2 \end{bmatrix} \quad (2.94)$$

This can be combined with (2.91) to yield

$$\begin{bmatrix} r_3 \\ r'_3 \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} r_1 \\ r'_1 \end{bmatrix} \quad (2.95)$$

Notice that optical elements in table 2.1 can be combined simply by 2x2 matrix multiplication. However, the multiplication is in reverse order. A similar result holds for beam matrices, but it is not as easy to show. As a clarifying example, note the matrix for a homogeneous medium of length d, and that

$$\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2d \\ 0 & 1 \end{bmatrix} \quad (2.96)$$

After propagating a distance  $d$  twice, the result must be the same as traveling a distance  $2d$ , and it is. The matrix theory agrees with common sense.

One important difference between the ray and beam matrix theories is that the ray theory demands that the matrix elements be real, while there is no such restriction on the beam theory. Hence, the complex lenslike media matrix in Table 2.1 applies to rays only in the special case of real  $A$ ,  $B$ ,  $C$ , and  $D$  matrix elements. The apodized (Gaussian) aperture does not, of course, exist for the ray theory.

Just as the Kogelnik transformation governs the propagation of the  $Q_x$  and  $Q_y$  parameters, the Casperson transformation (also known as the "AB law") governs the propagation of the  $S_x$  and  $S_y$  parameters [25]:

$$S_{x2} = \frac{S_{1x}}{A + B/q_{x1}} \quad (2.97)$$

Similarly, the phase parameter equation can be solved in terms of the matrix elements [26]. If  $\delta_\alpha \equiv A_\alpha + B_\alpha/q_{\alpha1}$ , then

$$P_2 - P_1 = -\frac{i}{2} \text{Re}[\ln \delta_x] + \frac{1}{2} \text{Im}[\ln \delta_x] - \frac{i}{2} \text{Re}[\ln \delta_y] + \frac{1}{2} \text{Im}[\ln \delta_y] \\ - \frac{S_{x1}^2}{2kn_{01}} \frac{B_x}{\delta_x} - \frac{S_{y1}^2}{2kn_{01}} \frac{B_y}{\delta_y} \quad (2.98)$$

where  $n_{01}$  is the index of refraction at the input of the optical element or system, and  $B_x$  (and  $B_y$ ) represent the  $B$  element of the  $ABCD$  matrix in the  $x$ -direction.

The matrix formalism is well suited to study resonators and other periodic structures. A matrix formula valid for unimodular  $2 \times 2$  matrices known as Sylvester's Theorem is useful when analyzing periodic and repetitive structures [1]:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^s = \frac{1}{\sin \theta} \begin{bmatrix} A \sin(s\theta) - \sin[(s-1)\theta] & B \sin(s\theta) \\ C \sin(s\theta) & D \sin(s\theta) - \sin[(s-1)\theta] \end{bmatrix} \quad (2.99)$$

where  $\cos\theta = \frac{1}{2}(A + D)$ . If  $\frac{1}{2}(A + D)$  is less than unity in absolute value, then  $\theta$  will be real, and easily calculated. Otherwise  $\theta$  will be imaginary, and can be calculated by using the Euler relation which will lead to

$$\theta = -i \ln \left[ \frac{A+D}{2} \pm \left[ \left( \frac{A+D}{2} \right)^2 - 1 \right]^{1/2} \right] \quad (2.100)$$

It should be noted that for nonunimodular 2x2 matrices the square root of the determinant can be factored out of the matrix and Sylvester's theorem can be used on the resulting unimodular matrix and its new coefficient separately.

One may also use Sylvester's theorem to study repetitive ray paths. One obtains repetitive ray paths if  $r_2 = r_1$  and  $r'_2 = r'_1$ . A sufficient condition for ray path repetition is that

$$\theta = 2\pi k/s \quad (2.101)$$

which may be restated as

$$\frac{A+D}{2} = \cos^{-1} \left[ \frac{2\pi}{s/k} \right] \quad (2.102)$$

where both  $k$  and  $s$  are integers, and  $s/k$  is the number of ray path repetitions. For further discussion, see [27] which also contains an interesting geometrical interpretation of  $k$  and  $s$ .

The spotsize and radius of curvature of a beam in a resonator can be found by an oscillation condition [9] that states that the fields repeat after a round trip. Hence  $q_1 = q_2$ . Applying this to the Kogelnik transformation (2.93) yields

$$\frac{1}{R_x} - i \frac{2}{kn_o w_x^2} = \frac{D-A}{2B} \pm i \frac{[1 - (A+D)^2/4]^{1/2}}{B} \quad (2.103)$$

and the radius of curvature and spotsize of the beam can be found by taking the real and imaginary parts of the right hand side. If the resonator contains astigmatic elements, then the  $ABCD$  matrix may be different for the  $x$  and  $y$  directions. Consider the resonator in fig 2-2. If  $f$  is the "focal length" of the curved mirror, then

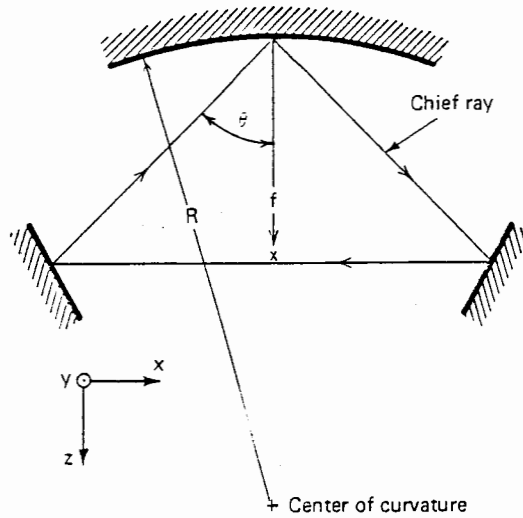


Figure 2-2. Astigmatic Laser Cavity [20].

$$f_x = f \cos \theta \quad (2.104)$$

$$f_y = f / \cos \theta \quad (2.105)$$

are the effective focal lengths in the  $x$  and  $y$  directions respectively. Such astigmatism is usually considered baneful and branded as distortion. However, sometimes astigmatic elements are added to a resonator to compensate for inherent astigmatism elements such as Brewster windows [28]. For more general misaligned systems, one must resort to a 3x3 matrix theory [29],[30]. Each ABCD matrix must be converted to the form

$$\begin{bmatrix} r_2 \\ r'_2 \\ 1 \end{bmatrix} = \begin{bmatrix} A & B & E \\ C & D & F \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r'_1 \\ 1 \end{bmatrix} \quad (2.106)$$

If the notation indicated in figure 2-3 is used, then

$$E = (1-A)\Delta_1 + (L - n_1 B)\Delta' \quad (2.107)$$

and

$$F = -C\Delta_1 + (n_2 - n_1 D)\Delta' \quad (2.108)$$

For systems with twists or image rotations there is an even more cumbersome 4x4

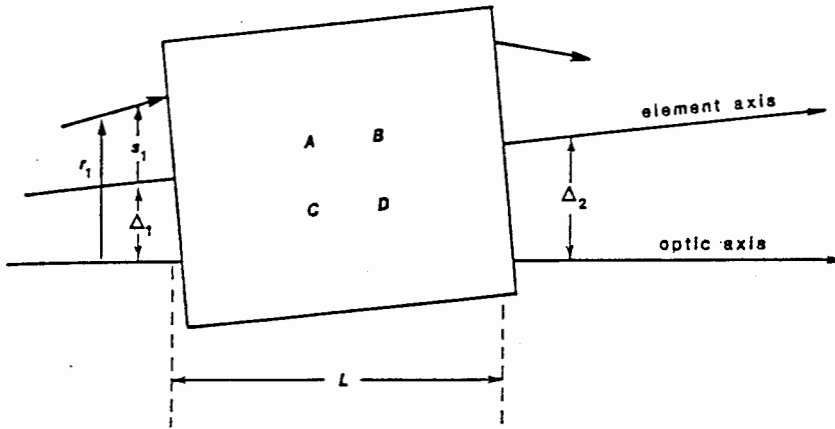


Figure 2-3. Notation for Analyzing a Misaligned Paraxial Element [29].

notation [29].

A confinement condition is implied from (2.103) since the spotsize,  $w$ , must be real and positive. In the special case of real elements, the lower sign must be chosen, yielding the confinement condition

$$-1 \leq \frac{1}{2}(A + D) \leq 1 \quad (2.109)$$

A confinement condition can be derived for light rays from Sylvester's theorem. If  $\theta$  is real then  $A$ ,  $B$ ,  $C$ , and  $D$  elements do not increase without bound. But this implies that  $-1 \leq \cos\theta \leq 1$ , which implies (2.109). Hence, the confinement condition for light rays is the same as that for beams in real media. An approximate stability condition for resonators can also be found by demanding that perturbations to  $1/q$  do not grow [25]

$$-1 \leq \frac{A + D}{2} \pm \left[ \left( \frac{A + D}{2} \right)^2 - 1 \right]^{1/2} \leq 1 \quad (2.110)$$

The sign to be chosen must be the same as that chosen in (2.102).

As mentioned earlier, experimental verification of a theory is its ultimate

test. The theory presented here applies to systems with gain. The predictions made by the theory agree well with experiment [18]. This study is primarily concerned with tapered media. Agreement with experiment can be made by the use of metal waveguides [31].

## SUMMARY

Consider a nonconductive, isotropic, magnetically linear and homogeneous, dielectrically linear (in a time invariant sense) material with no net unbalanced charge, and no absorption losses which exactly satisfies

$$P = C \cos(\omega t) + S \sin(\omega t)$$

and whose "plane wave" index of refraction satisfies

$$n = n_o - \frac{1}{2}[n_{2x}x^2 + n_{2y}y^2]$$

which is illuminated by a linearly polarized, sinusoidally time-varying, elliptic fundamental Gaussian beam whose major axis is parallel or perpendicular to the major axis of the index profile of the guiding structure. Furthermore, suppose that there are no external fields or temperature gradients.

In such a mythical material, if classical fields are used, the plane wave index varies negligibly in a wavelength, and  $n_o$  is much greater than the bracketed quantity above, then for nearly plane waves near the center of the material, the "propagating beam will remain Gaussian even though its spotsize, radius of curvature, amplitude, phase, and direction of propagation are significantly altered by the inhomogeneous medium" [21].

Introduced are two matrix theories that can be used to study the paraxial propagation of light in optical systems with lenses, dielectric boundaries, complex lenslike media, etc. The ray theory enables one to find position and slope, while

the Gaussian beam theory enables one to obtain beam spotsize and radius of curvature. Both theories enable one to study resonators and other periodic structures.

# CHAPTER III

## OFF-AXIS HIGHER ORDER MODES

### INTRODUCTION

It is clear that the fundamental mode solution presented in chapter II is not the most general solution possible. The purpose of this chapter is to investigate a more general solution to (2.61). It will then be shown that this more general solution, when combined with some boundary conditions, yields higher order modes. The results will be general enough to apply to tapered media.

### REDUCTION TO ORDINARY EQUATIONS

The purpose here is to solve for the off-axis propagation of such modes in tapered complex lenslike media including those with transverse spatial variations of gain or loss. Before continuing,  $Q_{xy}$  will be eliminated by setting it identically to zero. As mentioned previously, what this amounts to is the constraint that any "elliptical" input beam have a major axis that is parallel or perpendicular to the major axis of the elliptical index variation of the medium under question. By demanding that  $n_{xy}$  be constant, the axis of the index is constrained to not twist. Furthermore, as mentioned earlier, if the elliptical input beam is parallel or perpendicular to the major axis of the index variation of the medium,  $Q_{xy}$  will be identically zero. Equation (2.61) then becomes

$$\left[ \frac{\partial^2 B}{\partial x^2} - 2i (Q_x x + S_x) \frac{\partial B}{\partial x} \right] + \left[ \frac{\partial^2 B}{\partial y^2} - 2i (Q_y y + S_y) \frac{\partial B}{\partial y} \right] - 2ikn_o \frac{\partial B}{\partial z}$$



$$\begin{aligned}
& - \left\{ \left[ Q_x^2 + kn_o \frac{\partial Q_x}{\partial z} + k^2 n_o n_{2x} \right] x^2 + \left[ Q_y^2 + kn_o \frac{\partial Q_y}{\partial z} + k^2 n_o n_{2y} \right] y^2 \right. \\
& \quad + \left[ Q_x S_x + kn_o \frac{\partial S_x}{\partial z} + \frac{1}{2} k^2 n_o n_{1x} \right] 2x + \left[ Q_y S_y + kn_o \frac{\partial S_y}{\partial z} + \frac{1}{2} k^2 n_o n_{1y} \right] 2y \\
& \quad \left. + \left[ 2kn_o \frac{\partial P}{\partial z} + S_x^2 + S_y^2 + i (Q_x + Q_y) + ik \frac{\partial n_o}{\partial z} \right] \right\} B = 0
\end{aligned} \tag{3.1}$$

Upon close inspection of (3.1), one may notice its resemblance to the Hermite differential equation (see equation (3.24)). To increase this similarity, choose

$$[Q_x^2 + kn_o \frac{\partial Q_x}{\partial z} + k^2 n_o n_{2x}] = 0 \tag{3.2}$$

$$[Q_y^2 + kn_o \frac{\partial Q_y}{\partial z} + k^2 n_o n_{2y}] = 0 \tag{3.3}$$

$$[Q_x S_x + kn_o \frac{\partial S_x}{\partial z} + \frac{1}{2} k^2 n_o n_{1x}] = 0 \tag{3.4}$$

$$[Q_y S_y + kn_o \frac{\partial S_y}{\partial z} + \frac{1}{2} k^2 n_o n_{1y}] = 0 \tag{3.5}$$

$$[2kn_o \frac{\partial P}{\partial z} + S_x^2 + S_y^2 + i (Q_x + Q_y) + ik \frac{\partial n_o}{\partial z}] = -2 (ma_x^2 + na_y^2) \tag{3.6}$$

Notice that (3.2) - (3.5) is the same as (2.79) - (2.82). Hence all previous matrix results apply to the higher order modes as well! However, the phase parameter equation here (3.6) is slightly different from the one there (2.83). It should also be noted that for higher order modes, only the Gaussian factor responsible for beam displacement travels as the paraxial ray equation and (2.76)-(2.77). Another multiplicative complex Hermite factor propagates with its own center.

The resulting equation for  $B$  is

$$\begin{aligned}
& \left[ \frac{\partial^2 B}{\partial x^2} - 2i (Q_x x + S_x) \frac{\partial B}{\partial x} \right] + \left[ \frac{\partial^2 B}{\partial y^2} - 2i (Q_y y + S_y) \frac{\partial B}{\partial y} \right] - 2ikn_o \frac{\partial B}{\partial z} \\
& + 2 (ma_x^2 + na_y^2) B = 0
\end{aligned} \tag{3.7}$$

It is clear that separation of variables is in our future, but the  $z$ -dependence of the coefficients is menacing. What is needed is to introduce as many new choosable  $z$ -dependent functions as possible in an attempt to nullify those in (3.7). One is led to the change of variables

$$x' = a_x(z)x + b_x(z) \quad (3.8)$$

$$y' = a_y(z)y + b_y(z) \quad (3.9)$$

$$z' = z \quad (3.10)$$

If  $\alpha$  represents an unprimed cartesian coordinate then by definition

$$\frac{\partial}{\partial \alpha} = \frac{\partial x'}{\partial \alpha} \frac{\partial}{\partial x'} + \frac{\partial y'}{\partial \alpha} \frac{\partial}{\partial y'} + \frac{\partial z'}{\partial \alpha} \frac{\partial}{\partial z'} \quad (3.11)$$

from which it follows that

$$\frac{\partial}{\partial x} = a_x \frac{\partial}{\partial x'} \quad \frac{\partial^2}{\partial x^2} = a_x^2 \frac{\partial^2}{\partial x'^2} \quad (3.12)$$

$$\frac{\partial}{\partial y} = a_y \frac{\partial}{\partial y'} \quad \frac{\partial^2}{\partial y^2} = a_y^2 \frac{\partial^2}{\partial y'^2} \quad (3.13)$$

$$\frac{\partial}{\partial z} = \left[ \frac{\partial a_x}{\partial z} x + \frac{\partial b_x}{\partial z} \right] \frac{\partial}{\partial x'} + \left[ \frac{\partial a_y}{\partial z} y + \frac{\partial b_y}{\partial z} \right] \frac{\partial}{\partial y'} + \frac{\partial}{\partial z'} \quad (3.14)$$

Inserting change of variable equations (3.8) to (3.14) into equation (3.7) yields

$$\begin{aligned} & a_x^2 \frac{\partial^2 B}{\partial x'^2} - 2i \left[ \left( a_x Q_x + kn_o \frac{\partial a_x}{\partial z} \right) \left( \frac{x' - b_x}{a_x} \right) + \left( a_x S_x + kn_o \frac{\partial b_x}{\partial z} \right) \right] \frac{\partial B}{\partial x'} + \\ & a_y^2 \frac{\partial^2 B}{\partial y'^2} - 2i \left[ \left( a_y Q_y + kn_o \frac{\partial a_y}{\partial z} \right) \left( \frac{y' - b_y}{a_y} \right) + \left( a_y S_y + kn_o \frac{\partial b_y}{\partial z} \right) \right] \frac{\partial B}{\partial y'} \\ & - 2ikn_o \frac{\partial B}{\partial z'} + 2(ma_x^2 + na_y^2)B = 0 \end{aligned} \quad (3.15)$$

Now the coefficients can be forced "constant" if we choose

$$\left[ a_x Q_x + kn_o \frac{\partial a_x}{\partial z} \right] \left[ \frac{1}{a_x} \right] = -ia_x^2 \quad (3.16)$$

$$\left[ a_x Q_x + kn_o \frac{\partial a_x}{\partial z} \right] \left[ -\frac{b_x}{a_x} \right] + \left[ a_x S_x + kn_o \frac{\partial b_x}{\partial z} \right] = 0 \quad (3.17)$$

$$\left[ a_y Q_y + kn_o \frac{\partial a_y}{\partial z} \right] \left[ \frac{1}{a_y} \right] = -ia_y^2 \quad (3.18)$$

$$\left[ a_y Q_y + kn_o \frac{\partial a_y}{\partial z} \right] \left[ -\frac{b_y}{a_y} \right] + \left[ a_y S_y + kn_o \frac{\partial b_y}{\partial z} \right] = 0 \quad (3.19)$$

then

$$a_x^2 \left[ \frac{\partial^2 B}{\partial x'^2} - 2x' \frac{\partial B}{\partial x'} + 2mB \right] + a_y^2 \left[ \frac{\partial^2 B}{\partial y'^2} - 2y' \frac{\partial B}{\partial y'} + 2nB \right] - 2ikn_o \frac{\partial B}{\partial z'} = 0 \quad (3.20)$$

An important fact here is that  $m$  and  $n$  have not been chosen. Separating variables

in  $z$  would uselessly redefine them. Therefore choose

$$B(x', y', z') = C(x', y') \quad (3.21)$$

then

$$a_x^2 \left[ \frac{\partial^2 C}{\partial x'^2} - 2x' \frac{\partial C}{\partial x'} + 2mC \right] + a_y^2 \left[ \frac{\partial^2 C}{\partial y'^2} - 2y' \frac{\partial C}{\partial y'} + 2nC \right] = 0 \quad (3.22)$$

since  $a_x$  and  $a_y$  are the only functions of  $z$  in the equation, and they are in general not zero, the bracketed quantities must be exactly zero. However, this leads to two equations with only one unknown. One solution that satisfies both equations can easily be found through separation of variables:

$$C(x', y') = H_m(x') H_n(y') \quad (3.23)$$

$$\frac{\partial^2 H_m}{\partial x'^2} - 2x' \frac{\partial H_m}{\partial x'} + 2mH_m = 0 \quad (3.24)$$

$$\frac{\partial^2 H_n}{\partial y'^2} - 2y' \frac{\partial H_n}{\partial y'} + 2nH_n = 0 \quad (3.25)$$

If  $m$  and  $n$  are integers then  $H_m$  and  $H_n$  can be written in terms of elementary functions (in fact just polynomials). A whole family of polynomials can be defined by varying the integer  $m$  (or  $n$ ). The set of polynomials defined are known as the Hermite polynomials, and (3.24) is known as the Hermite differential equation.

The final solution for a single mode of the electric field is

$$\vec{E}' = \{ (E_{om,on} e^{-i\phi_{om,on}}) e^{-ik \int n_o dz} e^{-i[1/2 Q_x(z)x^2 + 1/2 Q_y(z)y^2 + Q_{xy}(z)xy + S_x(z)x + S_y(z)y + P(z)]} \times \quad (3.26)$$

$$H_m(a_x(z)x + b_x(z)) H_n(a_y(z)y + b_y(z)) \} \vec{i}_x$$

$$\vec{E} = \text{Re} \{ (E_{om,on} e^{-i\phi_{om,on}}) e^{-ik \int n_o dz} e^{-i[1/2 Q_x(z)x^2 + 1/2 Q_y(z)y^2 + Q_{xy}(z)xy + S_x(z)x + S_y(z)y + P(z)]} \times \quad (3.27)$$

$$H_m(a_x(z)x + b_x(z)) H_n(a_y(z)y + b_y(z)) e^{i\omega t} \} \vec{i}_x$$

$$= E_{om,on} (C_r^2 + C_i^2)^{1/2} e^{k \int n_{oi} dz + [1/2 Q_{xx}(x-d_{xa})^2 + 1/2 Q_{yy}(y-d_{ya})^2 - 1/2 Q_{xx} d_{xa}^2 - 1/2 Q_{yy} d_{ya}^2 + P_i]} \times \quad (3.28)$$

$$\cos \left[ 1/2 Q_{xx}(x-d_{xp})^2 + 1/2 Q_{yy}(y-d_{yp})^2 - 1/2 Q_{xx} d_{xp}^2 - 1/2 Q_{yy} d_{yp}^2 + P_r + k \int n_{or} dz - \tan^{-1}(C_i/C_r) - \omega t + \phi_{om,on} \right] \vec{i}_x$$

$$= E_{om,on} e^{k \int n_{oi} dz - 1/2 Q_{xx} d_{xa}^2 - 1/2 Q_{yy} d_{ya}^2 + P_i} \times \quad (3.29)$$

$$[(H_{mr}^2 + H_{mi}^2)^{1/2} e^{1/2 Q_{xx}(x-d_{xa})^2}] [(H_{nr}^2 + H_{ni}^2)^{1/2} e^{1/2 Q_{yy}(y-d_{ya})^2}]$$

$$\cos \left[ (P_r + k \int n_{or} dz - 1/2 Q_{xx} d_{xp}^2 - 1/2 Q_{yy} d_{yp}^2 + \phi_{om,on} - \omega t) - \right]$$

$$(\frac{1}{2}Q_{xr}(x-d_{xp})^2 + \tan^{-1}H_{mi}/H_{mr}) - (\frac{1}{2}Q_{yr}(y-d_{yp})^2 + \tan^{-1}H_{ni}/H_{nr})] \bar{i}_x$$

where again, the  $r$  and  $i$  subscripts denote real and imaginary parts, respectively (e.g.  $H_{mr}$  is the real part of  $H_m(a_x x + b_x)$ ). Since this was derived from a linear differential equation, one may sum up the solutions (or modes). Therefore,

$$\bar{E}_{total} = \sum_{m,n} \bar{E} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \bar{E} \quad (3.30)$$

The magnetic fields can be obtained from (2.23). These solutions are often referred to as the Hermite-Gaussian modes of complex argument. This refers to the fact that the arguments of the Hermite polynomials are complex. A similar set of modes can be derived by demanding that the argument of the Hermite polynomials be real. However, this can only be done under the constraint that the index profile have no spatial variations of gain or loss. These modes are referred to as the Hermite-Gaussian modes of real argument. It should be stressed that these two types of modes are independent in the sense that real argument modes are not a special case of complex argument modes, and vice-versa. In the special case that the index profile has circular symmetry, it may be simpler to use cylindrical rather than rectangular coordinates. In a completely analogous derivation, a new set of modes may be derived. Instead of Hermite polynomials in each of the cartesian directions, one obtains Laguerre polynomials in the radial direction. These modes are therefore known as the Laguerre-Gaussian modes, and they exist with both real and complex arguments as well [32]. It should be noted that the Laguerre-Gaussian modes of real argument have the same restriction of no spatial gain variation that the Hermite-Gaussian modes of real argument had.

The Hermite-Gaussian modes of real argument are orthogonal and form a complete set [33]. The Hermite-Gaussian modes of complex argument are biorthogonal and a complete set can be defined [34]. Hence, this analysis is

capable of handling input beam with arbitrary transverse spatial distributions since they can, theoretically, be expanded as an infinite sum of Hermite-Gaussian modes.

## SOLUTIONS OF THE BEAM EQUATIONS

Maxwell's equations have been reduced to ordinary differential equations for higher order modes, including those in media with transverse spatial variations of the gain or loss profile with off-axis input. These equations will be solved exactly. For clarity, the set of ordinary differential equations to be solved are rewritten here.

$$Q_x^2 + kn_o \frac{\partial Q_x}{\partial z} + k^2 n_o n_{2x} = 0 \quad (3.31)$$

$$Q_y^2 + kn_o \frac{\partial Q_y}{\partial z} + k^2 n_o n_{2y} = 0 \quad (3.32)$$

$$Q_x S_x + kn_o \frac{\partial S_x}{\partial z} + \frac{1}{2} k^2 n_o n_{1x} = 0 \quad (3.33)$$

$$Q_y S_y + kn_o \frac{\partial S_y}{\partial z} + \frac{1}{2} k^2 n_o n_{1y} = 0 \quad (3.34)$$

$$2kn_o \frac{\partial P}{\partial z} + S_x^2 + S_y^2 + i(Q_x + Q_y) + ik \frac{\partial n_o}{\partial z} = -2(ma_x^2 + na_y^2) \quad (3.35)$$

$$\left[ a_x Q_x + kn_o \frac{\partial a_x}{\partial z} \right] \left[ \frac{1}{a_x} \right] = -ia_x^2 \quad (3.36)$$

$$\left[ a_x Q_x + kn_o \frac{\partial a_x}{\partial z} \right] \left[ -\frac{b_x}{a_x} \right] + \left[ a_x S_x + kn_o \frac{\partial b_x}{\partial z} \right] = 0 \quad (3.37)$$

$$\left[ a_y Q_y + kn_o \frac{\partial a_y}{\partial z} \right] \left[ \frac{1}{a_y} \right] = -ia_y^2 \quad (3.38)$$

$$\left[ a_y Q_y + kn_o \frac{\partial a_y}{\partial z} \right] \left[ -\frac{b_y}{a_y} \right] + \left[ a_y S_y + kn_o \frac{\partial b_y}{\partial z} \right] = 0 \quad (3.39)$$

These are nine coupled ordinary differential equations with nine unknowns.

The beam parameter,  $Q_x$ , can be solved for independently from (3.31). Similarly,  $Q_y$  can be solved via (3.32). However, for an arbitrary  $n_2(z)$ , these equations

cannot be solved. Note that each of the other equations contain at least one of the two beam parameters. Hence, none of these equations can, in general, be solved. Chapter IV is dedicated to finding solutions to (3.31) for different possible variations of  $n_2(z)$ . The purpose here is to demonstrate that the other seven equations can be solved for in terms of the two beam parameters. Hence, for purposes of this chapter, (3.31) and (3.32) are considered to be solved.

For specificity,  $n_o(z)$  will be constrained to be constant, and  $n_{1x}(z)$  and  $n_{1y}(z)$  will be constrained to be zero. This amounts to "unbent" media whose complex index is constant on the waveguide axis. Some of the equations are very similar to others. This essentially allows one to solve only half of the equations. If  $\alpha$  represents either  $x$  or  $y$ , then

$$Q_\alpha S_\alpha + kn_o \frac{\partial S_\alpha}{\partial z} = 0 \quad (3.40)$$

$$a_\alpha Q_\alpha + kn_o \frac{\partial a_\alpha}{\partial z} + ia_\alpha^3 = 0 \quad (3.41)$$

$$\frac{\partial b_\alpha}{\partial z} - \frac{1}{kn_o a_\alpha} \left[ a_\alpha Q_\alpha + kn_o \frac{\partial a_\alpha}{\partial z} \right] b_\alpha = -\frac{S_\alpha a_\alpha}{kn_o} \quad (3.42)$$

$$\frac{\partial P}{\partial z} = -\frac{ma_x^2 + na_y^2}{kn_o} - \frac{S_x^2 + S_y^2}{2kn_o} - i \frac{Q_x + Q_y}{2kn_o} \quad (3.43)$$

If  $r_\alpha$  is defined by

$$\frac{Q_\alpha}{kn_o} \equiv \frac{1}{r_\alpha} \frac{\partial r_\alpha}{\partial z} = \frac{\partial}{\partial z} (\ln r_\alpha) \quad (3.44)$$

then (3.40) - (3.43) become

$$\frac{1}{r_\alpha} \frac{\partial r_\alpha}{\partial z} S_\alpha + \frac{\partial S_\alpha}{\partial z} = 0 \quad (3.45)$$

$$\frac{a_\alpha^2}{r_\alpha} \frac{\partial r_\alpha}{\partial z} + \frac{1}{2} \frac{\partial a_\alpha^2}{\partial z} + \frac{i}{kn_o} (a_\alpha^2)^2 = 0 \quad (3.46)$$

$$\frac{\partial b_\alpha}{\partial z} - \left[ \frac{1}{r_\alpha} \frac{\partial r_\alpha}{\partial z} + \frac{1}{a_\alpha} \frac{\partial a_\alpha}{\partial z} \right] b_\alpha = -\frac{S_\alpha a_\alpha}{kn_o} \quad (3.47)$$

$$\frac{\partial P}{\partial z} = -\frac{ma_x^2 + na_y^2}{kn_o} - \frac{S_x^2 + S_y^2}{2kn_o} - \frac{i}{2} \left[ \frac{1}{r_x} \frac{\partial r_x}{\partial z} + \frac{1}{r_y} \frac{\partial r_y}{\partial z} \right] \quad (3.48)$$

The first equation, (3.45), is separable. The solution is

$$S_\alpha = S_{o\alpha} r_\alpha^{-1} \quad (3.49)$$

Similarly, (3.47) has a well-known "solution":

$$b_\alpha = -\frac{S_{o\alpha} a_\alpha r_\alpha}{kn_o} \left[ \int r_\alpha^{-2} dz + c_{2\alpha} \right] \quad (3.50)$$

The phase parameter equation becomes

$$P - P_o = \frac{-1}{2kn_o} \left[ 2 \int (ma_x^2 + na_y^2) dz + S_{ox}^2 \int r_x^{-2} dz + S_{oy}^2 \int r_y^{-2} dz \right] - \frac{i}{2} \ln r_x r_y \quad (3.51)$$

These last two equations are in some sense solved if we knew  $a_\alpha$ . Time must be taken to solve (3.46). For illustrative purposes, the author will demonstrate the method he used to obtain solutions. Equation (3.46) can be rewritten

$$(a_\alpha^2)^2 - \frac{ikn_o}{r_\alpha} \frac{\partial r_\alpha}{\partial z} a_\alpha^2 - \frac{ikn_o}{2} \frac{\partial a_\alpha^2}{\partial z} = 0 \quad (3.52)$$

A prudent substitution (and an obvious one to make considering the mathematical methods developed in the next chapter) is

$$a_\alpha^2 = \frac{ikn_o}{2} \left[ -Q_{2\alpha} + \frac{1}{r_\alpha} \frac{\partial r_\alpha}{\partial z} \right] \quad (3.53)$$

which yields a new equation

$$Q_{2\alpha}^2 + \frac{\partial Q_{2\alpha}}{\partial z} - \left[ \left( \frac{1}{r_\alpha} \frac{\partial r_\alpha}{\partial z} \right)^2 + \frac{\partial}{\partial z} \left( \frac{1}{r_\alpha} \frac{\partial r_\alpha}{\partial z} \right) \right] = 0 \quad (3.54)$$

An obvious solution to this is

$$Q_{2\alpha} = \frac{1}{r_\alpha} \frac{\partial r_\alpha}{\partial z} \quad (3.55)$$

However, when substituted back into (3.53), only the trivial solution is obtained. Again, as shown in chapter IV, it is not difficult to obtain a more general solution to a Riccati equation (3.54) when one solution is known. The more general solution is

$$Q_{2\alpha} = \frac{1}{r_\alpha} \frac{\partial r_\alpha}{\partial z} + \frac{r_\alpha^{-2}}{c_{3\alpha} + \int r_\alpha^{-2} dz} \quad (3.56)$$

The validity of this result can easily be checked by direct substitution of (3.56) into (3.54). Our interest here is in  $a_\alpha$ . Therefore, substituting (3.56) back into (3.53) yields

$$a_\alpha^2 = -\frac{ikn_o}{2} \left[ \frac{r_\alpha^{-2}}{c_{3\alpha} + \int r_\alpha^{-2} dz} \right] \quad (3.57)$$

or

$$a_\alpha = \pm r_\alpha^{-1} \left[ \frac{-ikn_o/2}{c_{3\alpha} + \int r_\alpha^{-2} dz} \right]^{1/2} \quad (3.58)$$

Therefore,

$$b_\alpha = \mp \frac{S_{o\alpha}}{kn_o} \left[ \frac{-ikn_o/2}{c_{3\alpha} + \int r_\alpha^{-2} dz} \right]^{1/2} [c_{2\alpha} + \int r_\alpha^{-2} dz] \quad (3.59)$$

Similarly the phase parameter equation (3.51) can now be solved. However, before doing so, the integral should be solved, if possible. If one notes that  $C \equiv dA/dz$ ,  $D \equiv dB/dz$ ,  $AD - BC = 1$ , and that  $r = A + B/q_o$ , then it can be shown [26] by simple differentiation that

$$\int (A + B/q_{o\alpha})^{-2} dz = \frac{B}{A + B/q_{o\alpha}} \quad (3.60)$$

From which it follows that

$$S_\alpha = \frac{S_{o\alpha}}{A_\alpha + B_\alpha/q_{o\alpha}} \quad (3.61)$$

$$a_\alpha = \pm \frac{1}{A_\alpha + B_\alpha/q_{o\alpha}} \left[ \frac{-ikn_o/2}{c_{3\alpha} + B/(A_\alpha + B_\alpha/q_{o\alpha})} \right]^{1/2} \quad (3.62)$$

$$b_\alpha = \mp \frac{S_{o\alpha}}{kn_o} \left[ \frac{-ikn_o/2}{c_{3\alpha} + B/(A_\alpha + B_\alpha/q_{o\alpha})} \right]^{1/2} [c_{2\alpha} + B/(A_\alpha + B_\alpha/q_{o\alpha})] \quad (3.63)$$

$$P_2 - P_1 = \frac{i}{2} \left[ m \operatorname{Ln}(c_{3x} + B_x/(A_x + B_x/q_{ox})) + n \operatorname{Ln}(c_{3y} + B_y/(A_y + B_y/q_{oy})) \right] \quad (3.64)$$

$$- \frac{1}{2kn_o} \left[ \frac{S_{ox}^2 B_x}{A_x + B_x/q_{ox}} + \frac{S_{oy}^2 B_y}{A_y + B_y/q_{oy}} \right] - \frac{i}{2} \operatorname{Ln}(A_x + B_x/q_{ox}) - \frac{i}{2} \operatorname{Ln}(A_y + B_y/q_{oy})$$

Capital letters are used on the logarithms to remind the reader that they have com-



plex arguments [35]. These four equations are the principal result of the section.

## HIGHER ORDER MODE TRANSFORMATIONS

The Kogelnik transformation (ABCD law) governs the propagation of the two beam parameters,  $Q_x$  and  $Q_y$ .

$$\frac{Q_\alpha}{kn_o} = \frac{1}{q_\alpha} = \frac{C + D/q_{o\alpha}}{A + B/q_{o\alpha}} \quad (3.65)$$

This transformation is important because, for fundamental mode propagation, it has the following properties:

- i) The beam parameters yield physically important parameters. In this case, beam "spotsize" and phase front curvature can be easily garnered through the relationship

$$\frac{Q_\alpha}{kn_o} = \frac{1}{q_\alpha} = \frac{1}{R_\alpha} - i \frac{2}{kn_o w_\alpha^2} \quad (3.66)$$

- ii) The transformation is general enough so as to include boundary condition information (such as a thin lens).
- iii) Elements can be combined by the use of simple matrix multiplication
- iv) The transformation can be used in a wide variety of applications including resonator theory.

The Casperson transformation

$$S_\alpha = \frac{S_{o\alpha}}{A_\alpha + B_\alpha/q_{o\alpha}} \quad (3.67)$$

satisfies property i) since

$$d_{\alpha a} = -S_{\alpha i}/Q_{\alpha i} \quad d_{\alpha p} = -S_{\alpha r}/Q_{\alpha r} \quad (3.68)$$

where  $d_{\alpha a}$  is the displacement of the beam amplitude, and  $d_{\alpha p}$  is the displacement of the beam phase. The Casperson transformation also has the second and third

properties. The oscillation condition does not demand that resonators have beam displacements that repeat after a round trip. However, this transformation can be used to study resonator stability [25], and hence property iv) is satisfied.

The difficulty in studying higher order modes is not that these two previous transformations are invalid, however they contain only part of the information needed to obtain spotsize, phase curvature, and amplitude and phase displacement. One of the problems is that the beam no longer has one "spot" so that it no longer has one "spotsize". Similarly, the phase fronts are no longer parabolic (or spherical), so that phase front radius of curvature has less meaning. The purpose here is to develop two new transformations and demonstrate that they contain the above four properties.

To obtain the two new transformations, just apply boundary conditions to (3.62) and (3.63). After passing through freespace for a length  $z$ , then  $a(z) = a_{o\alpha}$  if  $z \rightarrow 0$  i.e. the value will be the initial value. Mathematically, the boundary conditions can be stated as follows:

$$a(A = 1, B \rightarrow 0) = a_{o\alpha} \quad (3.69)$$

Similar results apply to the  $b$  parameter. Now, (3.62) and (3.63) can be rewritten in "transformation form":

$$a_\alpha = \frac{a_{o\alpha}}{A_\alpha + B_\alpha/q_{o\alpha}} \left[ 1 + \frac{2ia_{o\alpha}^2}{kn_o} \frac{B_\alpha}{A_\alpha + B_\alpha/q_{o\alpha}} \right]^{-1/2} \quad (3.70)$$

$$b_\alpha = \left[ b_{o\alpha} - \frac{a_{o\alpha}S_{o\alpha}}{kn_o} \frac{B_\alpha}{A_\alpha + B_\alpha/q_{o\alpha}} \right] \left[ 1 + \frac{2ia_{o\alpha}^2}{kn_o} \frac{B_\alpha}{A_\alpha + B_\alpha/q_{o\alpha}} \right]^{-1/2} \quad (3.71)$$

In terms of these new parameters, the phase parameter equation can now be written

$$\begin{aligned} P - P_o = & \frac{i}{2} \left[ m \operatorname{Ln} \left( \frac{-ikn_o/2}{a_{ox}^2} + B_x/(A_x + B_x/q_{ox}) \right) + n \operatorname{Ln} \left( \frac{-ikn_o/2}{a_{oy}^2} + B_y/(A_y + B_y/q_{oy}) \right) \right] \\ & - \frac{1}{2kn_o} \left[ \frac{S_{ox}^2 B_x}{A_x + B_x/q_{ox}} + \frac{S_{oy}^2 B_y}{A_y + B_y/q_{oy}} \right] - \frac{i}{2} \operatorname{Ln}(A_x + B_x/q_{ox}) - \frac{i}{2} \operatorname{Ln}(A_y + B_y/q_{oy}) \end{aligned}$$

(3.72)

These equations clearly yield physically important parameters. Therefore, property i) is satisfied. However, their analytical relationship to these parameters is not obvious. An investigation into the physical significance of these two new parameters is the purpose of the next section. In the same way that the Casperson transformation applies to boundary condition information, so do the  $a$  and  $b$  parameters. Therefore, property ii) is satisfied. Though it is a long and tiring exercise in elementary arithmetic, property iii) can be shown to be valid. The oscillation condition can be applied to the  $a$  parameter, and addition stability criteria [25] can easily be applied to both transformations. Hence, property iv) is also valid.

### INTERPRETATION OF THE $a$ AND $b$ PARAMETERS

The purpose of this section is to examine the relationship between the  $a$  and  $b$  parameters, and physically interesting parameters (e.g. beam displacement).

The primary method employed here to learn about higher order modes will be to consider an example. The simplest higher order modes to analyze are the  $TEM_{0,1}$  and  $TEM_{1,0}$  modes. For specificity the  $TEM_{1,0}$  mode is considered here. However, the  $TEM_{0,1}$  mode can be obtained simply by interchanging the transverse coordinates. From (3.23)

$$\begin{aligned}
 C(x,y) &= H_1(a_x x + b_x) H_0(a_y y + b_y) \\
 &= H_1(a_x x + b_x) \\
 &= 2(a_x x + b_x) \\
 &= 2(a_{xr} x + b_{xr}) + 2i(a_{xi} x + b_{xi})
 \end{aligned} \tag{3.73}$$

where, as before, the  $r$  and  $i$  subscripts denote the real and imaginary parts, respectively. Of particular interest here is the magnitude of the field as opposed to its phase. It can be seen from (3.29) that it is necessary to calculate  $(H_{mr}^2 + H_{mi}^2)^{1/2}$ :

$$\begin{aligned}
(H_{mr}^2 + H_{mi}^2)^{1/2} &= 2[(a_{xr}x + b_{xr})^2 + (a_{xi}x + b_{xi})^2]^{1/2} \\
&= 2[(a_{xr}^2 + a_{xi}^2)x^2 + 2(a_{xr}b_{xr} + a_{xi}b_{xi})x + (b_{xr}^2 + b_{xi}^2)]^{1/2} \\
&= 2(a_{xr}^2 + a_{xi}^2)^{1/2} \left[ \left[ x + \frac{a_{xr}b_{xr} + a_{xi}b_{xi}}{a_{xr}^2 + a_{xi}^2} \right]^2 + \frac{b_{xr}^2 + b_{xi}^2}{a_{xr}^2 + a_{xi}^2} - \left[ \frac{a_{xr}b_{xr} + a_{xi}b_{xi}}{a_{xr}^2 + a_{xi}^2} \right]^2 \right] \\
&= 2(a_{xr}^2 + a_{xi}^2)^{1/2} \left[ \left[ x + \frac{a_{xr}b_{xr} + a_{xi}b_{xi}}{a_{xr}^2 + a_{xi}^2} \right]^2 + \left[ \frac{a_{xr}b_{xi} - a_{xi}b_{xr}}{a_{xr}^2 + a_{xi}^2} \right]^2 \right]^{1/2} \quad (3.74)
\end{aligned}$$

The intensity of the beam goes like the magnitude of the beam squared. Therefore,

$$\begin{aligned}
\langle I \rangle &= I_{om,on}(a_{xr}^2 + a_{xi}^2) \left[ \left[ x + \frac{a_{xr}b_{xr} + a_{xi}b_{xi}}{a_{xr}^2 + a_{xi}^2} \right]^2 + \left[ \frac{a_{xr}b_{xi} - a_{xi}b_{xr}}{a_{xr}^2 + a_{xi}^2} \right]^2 \right] \times \\
&\quad e^{2k \int n_{oi} dz + [Q_{xi}(x-d_{xi})^2 + Q_{yi}(y-d_{yi})^2 - Q_{xi}d_{xi}^2 - Q_{yi}d_{yi}^2 + 2P_i]} \quad (3.75)
\end{aligned}$$

where  $P_i$  is the imaginary part of (3.71). This result suggests that there are two symmetric functions that each have their own "center" which travel more or less independently. If the centers of each of these two functions do not coincide, the mode will be asymmetric. The center of the Gaussian factor will be as in (3.67).

The center of the polynomial factor is

$$d_{xa2} = -\frac{a_{xr}b_{xr} + a_{xi}b_{xi}}{a_{xr}^2 + a_{xi}^2} \quad (3.76)$$

As special cases, consider on-axis beams

$$\langle I \rangle_{on-axis} = I_{om,on}(b_{xr}^2 + b_{xi}^2) e^{2k \int n_{oi} dz + 2P_i} \quad (3.77)$$

and the center of an off-axis beam

$$\langle I \rangle_{beam\ center} = I_{om,on} \frac{(a_{xr}b_{xi} - a_{xi}b_{xr})^2}{a_{xr}^2 + a_{xi}^2} e^{2k \int n_{oi} dz - Q_{xi}d_{xi}^2 - Q_{yi}d_{yi}^2 + 2P_i} \quad (3.78)$$

## SUMMARY

Exact analytic formulas governing the propagation of off-axis multimode Gaussian light beams in elliptical complex lenslike media have been found. The analysis is general enough to include both tapered media and those with transverse spatial gain or loss variation. Two new transformations have been developed to

account for the higher order mode behavior. Thus all parameters have been stated in terms of the  $ABCD$  matrix formalism without the need for further integration.

The difficulty that arises is that the complex magnitude and phase of the Hermite polynomials are not always tractable. Unfortunately, this leads to a difficulty in interpreting the new parameters. However, one of the parameters has been shown to be related to beam displacement. If, for a particular application, transverse spatial variations off the gain or loss are not important, one may resort to the Hermite Gaussian modes of real argument for simplicity [32].

# CHAPTER IV

## SOLUTIONS FOR TAPERED MEDIA

### INTRODUCTION

Through the approximations and constraints emphasized in the summary of the second chapter, Maxwell's equations were reduced to a set of coupled ordinary differential equations. For fundamental mode propagation in a quadratic index profile which does not vary longitudinally, these equations were solved. Recently there has been interest in media whose index profiles are longitudinally varying (i.e. tapered). Applications include uses such as power concentration, image minification, and coupling [36]. Applications is the subject of the next chapter.

To solve for propagation in such a tapered medium, one must only solve the following equations with the appropriate boundary conditions.

$$Q_x^2 + kn_o \frac{\partial Q_x}{\partial z} + k^2 n_o n_{2x}(z) = 0 \quad (4.1)$$

$$Q_y^2 + kn_o \frac{\partial Q_y}{\partial z} + k^2 n_o n_{2y}(z) = 0 \quad (4.2)$$

To obtain the spotsize and radius of curvature of the propagating beam, recall that, for the fundamental mode only

$$\frac{Q_x}{kn_o} = \frac{1}{R_x} - i \frac{2}{kn_o w_x^2} \quad (4.3)$$

To obtain the beam displacement in the  $x$  direction, recall that

$$d_{xa} = -S_{xi}/Q_{xi} \quad (4.4)$$

Similar formulas apply for the  $y$  direction. Notice that (4.1) and (4.2) can be

solved independent of each other. Note also that they are functionally equivalent. In the next section it will be shown that (4.1) can be transformed to a linear second-order homogeneous equation.

The primary interest in this chapter lies in on-axis quadratic index profiles of the form

$$n = n_o - n_2(z) (x^2 + y^2)/2 \quad (4.5)$$

$$= n_o - n_2(z) r^2/2 \quad (4.6)$$

If  $n_{cl}$  denotes the index of the "cladding" and  $r_{cl}$  is the radius of the "cladding", then

$$n_{cl} = n_o - n_2(z) r_{cl}^2(z)/2 \quad (4.7)$$

It is clear that along surfaces of constant index,  $r_{cl}(z)$  varies in  $z$  if  $n_2(z)$  does. This variation of the fiber radius is known as tapering. Hence,  $r_{cl}(z)$  shall be referred to as the taper function. There exists analogous definitions when astigmatic tapers are involved. Along the  $y$ -axis,  $y = 0$ , and (4.5) will then imply that

$$n_{cl} = n_o - n_{2x}(z) x_{cl}^2(z)/2 \quad (4.8)$$

Similar results hold on the  $x$ -axis.

## ALTERNATE FORMS OF THE HILL-TYPE EQUATION

Of the many types of equations in applied mathematics and physics, perhaps most pervasive is the ordinary, linear, second-order, homogeneous, non-constant coefficient, differential equation. One reason for this is that more complicated equations are, in general, more difficult to solve. Therefore, clever substitutions and/or approximations are invoked to reduce the more complicated equation to this simpler form. It is the intent of this section to demonstrate the existence of other common forms of this equation. In its most general form, the aforementioned equation is

$$y'' + a_1(z)y' + a_2(z)y = 0 \quad (4.9)$$

It will be referred to as the standard form. The primes indicate differentiation with respect to  $z$ .

### Hill Form

Though it is somewhat surprising, it is not difficult to show that (4.9) can be put into the form

$$y'' + b(z)y = 0 \quad (4.10)$$

by the substitution [37]

$$r = ye^{\frac{1}{2} \int a_1(z) dz} \quad (4.11)$$

where

$$b(z) = -\frac{1}{2}a_1' - \frac{1}{4}a_1^2 + a_2 \quad (4.12)$$

Hence, (4.10) can be considered an alternate form of (4.9). In the special case of periodic  $b(z)$ , (4.10) is known as the Hill equation. For our purposes here,  $b(z)$  may or may not be periodic. Hence (4.10) shall be referred to as the Hill-type equation, or the Hill form of (4.9).

### Riccati Form

Similarly, it can be shown that the equation

$$Q^2 + k_o(z)Q' + k_o(z)k_2(z) = 0 \quad (4.13)$$

can be put in standard form, and hence Hill form. If

$$Q = (k_o r')/r = k_o \frac{d}{dz}(\ln r) \quad (4.14)$$

then it follows that

$$(k_o r')' + k_2 r = 0 \quad (4.15)$$

which is of standard form. Equation (4.13) is known as the Riccati equation. The



importance of this is not subtle. Nonlinear, non-constant coefficient, inhomogeneous differential equations are difficult to solve. The fact that a simple transformation turns it into a linear homogeneous equation is to say the least unexpected.

### Generalized Riccati Form

The equation

$$Q^2 + f_1(z)Q + f_2(z)Q' + f_3(z) = 0 \quad (4.16)$$

is known as the generalized Riccati equation. The substitution

$$Q = f_2 Q_2 / \beta - \frac{1}{2}(f_1 + f_2') \quad (4.17)$$

where  $\beta$  is a constant, yields a Riccati equation

$$Q_2^2 + \beta Q_2' + \beta^2 \left[ \frac{1}{4} \frac{f_2'^2 - f_1^2}{f_2^2} - \frac{(f_1' + f_2'')}{2f_2} + \frac{f_3}{f_2^2} \right] = 0 \quad (4.18)$$

Note that this is actually a special case of the Riccati equation since  $\beta$  is constant here. In the special case that  $f_1(z) = 0$ , substitution (4.17) demonstrates a way to transform  $f_2(z)$  to a constant. This is an alternative to another more complicated method previously suggested [23].

It has been shown that the standard form can be put into Hill form. The Riccati form can be put into the standard form, and therefore the Hill form. The generalized Riccati form can be put into the Riccati form, which can be put into the standard form, which can be put into the Hill form. Hence, the Generalized Riccati equation, the Riccati equation, the standard form equation, and the Hill type equation can all be viewed as different forms of the same equation. Whichever form is chosen is therefore a matter of personal preference. For purposes of this study, the Hill form will be the form of choice.

## GENERATING SOLUTIONS

The Hill-type equation cannot be solved with arbitrary nonconstant coefficients. Therefore, neither can any of the other forms, in general, be solved. It is the purpose of this section to demonstrate a way of generating solutions to the Hill-type equation. The fundamental idea behind generating solutions, is the simplistic philosophy of "assume a solution, and see what you get" (for coefficients).

Generating solutions to the Hill-type equation is particularly simple since it can be rewritten as

$$f(z) = -r''/r \quad (4.19)$$

Generating solutions to the Riccati equation is also not difficult since the coefficient to the derivative can always be transformed to unity. In this case, the Riccati equation can be rewritten as

$$f(z) = -(Q^2 + Q') \quad (4.20)$$

Often, there exists a simple solution to the equation

$$Q^2 + Q' + f(z) = 0 \quad (4.21)$$

However, this equation is nonlinear, and hence one cannot merely add solutions (i.e. superposition does not hold) [38]. As an example, consider (4.21) with  $f(z) = -1$ . Even the casual observer will notice that  $Q = 1$  is a solution. However, this of course is not in any sense general. One thing we do know is that solutions to (4.21) must have an integration constant, and our solution does not. However, there exists a simple method for finding a more general solution to (4.21) given any nonzero solution. Consider

$$Q = q + 1/v \quad (4.22)$$

where  $q$  is the known solution. Inserting (4.22) into (4.21), and using the fact that  $q$  is a solution, yields

$$v' - 2qv = 1 \quad (4.23)$$

The new function  $v$  can be solved for in terms of the known function  $q$  since this differential equation is linear and of first order. Solving for  $v$  and inserting the result into (4.22) gives

$$Q = q + \frac{e^{-2\int q dz}}{c + \int e^{-2\int q dz} dz} \quad (4.24)$$

It is important to notice that (4.24) contains the required integration constant. If  $Q = r'/r$  then it is not difficult to solve for  $r$ :

$$r = \left[ c_1 + \int e^{-2\int q dz} dz \right] e^{\int q dz} \quad (4.25)$$

It should be clear that  $r$  is a solution to (4.10). This is desirable in that the  $ABCD$  formalism introduced in chapter II is best suited for solutions of (4.10).

Similarly, in the case that if one solution to (4.10) is known, the general solution can be found by an entirely analogous method. Consider

$$r_2 = r_1 \int v dz \quad (4.26)$$

where  $r_1$  is a known solution. Substituting (4.26) into (4.10), and using the fact that  $r_1$  is a solution implies that  $v = 1/r_1^2$ . Hence

$$r_2 = r_1 \int \frac{1}{r_1^2} dz \quad (4.27)$$

However, the Wronskian of  $r_1$  and  $r_2$  is unity. Hence, they are linearly independent solutions, and the general solution is

$$r = r_1(a + b \int 1/r_1^2 dz) \quad (4.28)$$

As an example, consider  $b(z) = 1$  in (4.10). Of course  $\sin(z)$  is one of the solutions. From (4.27), the other linearly independent solution is

$$\begin{aligned}
 r_2 &= \sin(z) \int 1/\sin^2(z) dz \\
 &= \sin(z)[-cot(z)] \\
 &= -\cos(z)
 \end{aligned}
 \tag{4.29}$$

of course. It should also be noted that if

$$r_1 = e^{\int q_1 dz} \tag{4.30}$$

then (4.28) and (4.25) are, as expected, identical within a multiplicative integration constant.

There has been recent interest in solutions to

$$r'' + (f(z) + g(z))r = 0 \tag{4.31}$$

where the solution to

$$u'' + f(z)u = 0 \tag{4.32}$$

is considered to be known. The interest in such solutions was fueled by a specific solution found by Casperson [39]:

$$\frac{d^2 r}{dz^2} + \left[ \frac{F}{[1 + G\cos(2z)]^4} + \frac{4G\cos(2z)}{1 + G\cos(2z)} \right] r = 0 \tag{4.33}$$

The equation has a variety of applications. Interestingly,  $r(z)$  can be expressed solely in terms of elementary functions. It was first shown by Wu and Shih [40] that (4.33) could be solved by a simple construction method. They also introduced a few new, but similar solutions. Renne [41] pointed out that similar constructions techniques have been long known. Takayama [42] points out that (4.33) and the Wu and Shih solutions contains singularities, and demonstrate a method of constructing singularity-free solutions. Nassar and Machado [43] submit a generalized theory to construct solutions of

$$r'' + (f_1(z) + f_2(z) + f_3(z) + \dots)r = 0 \tag{4.34}$$

These papers share the point of view that (4.33) is just one in a whole class of solvable Hill-type equations. The purpose here is to show that this construction

technique is only one in a whole class of construction techniques.

As mentioned above, the basic idea behind construction techniques (generating solutions) is basically to assume a solution, and see what coefficient you get. With this in mind, assume a solution of the form

$$r(z) = u(z)e^{i\int v(z)dz} \quad (4.35)$$

It immediately follows that

$$r'' - \left[ \frac{u''}{u} + 2i\frac{u'}{u}v - v^2 + iv' \right] r = 0 \quad (4.36)$$

which is of the form (4.31) as long as the solution to (4.32) is considered to be known. Each class of construction techniques will be special cases of (4.36).

The coefficient in (4.36) has four terms. In the special case that the second and fourth terms cancel,  $v(z)$  can be solved for:

$$v(z) = F^{1/2}/u^2 \quad (4.37)$$

then

$$r'' - \left[ \frac{u''}{u} - \frac{F}{u^4} \right] r = 0 \quad (4.38)$$

This is the fundamental idea behind the Wu and Shih paper and the ones that followed. Only in this special case is the other linearly independent solution found easily. It is

$$r(z) = u(z)e^{-i\int v(z)dz} \quad (4.39)$$

where  $v(z)$  is chosen by (4.37). In general, the other linearly independent solution can be found from (4.27):

$$r_2 = ue^{i\int v(z)dz} \int \frac{e^{-2i\int v(z)dz}}{u^2} dz \quad (4.40)$$

In the special case that  $b(z)$  in (4.10) is real and  $r_1$  is complex (nonreal), then, taking the complex conjugate of (4.10), it can be seen that  $r_1^*$  is also a solution.

As an example of the applicability of (4.38), consider  $u = 1 + G \cos(2z)$ . It is not difficult to see that (4.38) becomes identical to (4.33), the Caspersen solution. Since  $u$  is known,  $v$  is known from (4.37), and  $r(z)$  becomes known from (4.35) and (4.39).

Renne [41] pointed out the (4.38) exists in another form. Indeed, both that form, and (4.38) are specific examples of a more general form that can be garnered by defining

$$u \equiv w^m \quad (4.41)$$

In this case, (4.38) can be rewritten

$$r'' - \left[ m \frac{w''}{w} + m(m-1) \left( \frac{w'}{w} \right)^2 - \frac{F}{w^{4m}} \right] r = 0 \quad (4.42)$$

If  $m = 1$ , then (4.42) reduces to (4.38) which can be a useful form. Renne's form [41] corresponds to  $m = 1/2$ . Another important special case is  $m = -1/4$ . Then

$$r'' + \left[ \frac{1}{4} \frac{w''}{w} - 5/16 \left( \frac{w'}{w} \right)^2 + Fw \right] r = 0 \quad (4.43)$$

This is interesting since one can just pick  $w$  without knowledge of the solution  $y$  where

$$y'' + w(z)y = 0 \quad (4.44)$$

Now consider another class of generatable solutions. Suppose the second and third terms in (4.36) add to zero. This happens when

$$v = 2iu'/u \quad (4.45)$$

then

$$r'' + \left[ \frac{u''}{u} - 2 \left( \frac{u'}{u} \right)^2 \right] r = 0 \quad (4.46)$$

Note that the  $u''/u$  term has a different sign than that in (4.38). Substitution (4.41) leads to

$$r'' + m \left[ \frac{w''}{w} - (m+1) \left( \frac{w'}{w} \right)^2 \right] r = 0 \quad (4.47)$$

Similarly, choose the third and fourth terms in (4.36) to cancel. This implies that

$$v = \frac{-i}{\alpha + z} \quad (4.48)$$

Therefore,

$$r'' - \left[ \frac{u''}{u} + \frac{2}{\alpha + z} \frac{u'}{u} \right] r = 0 \quad (4.49)$$

Again, substitution (4.41) yields

$$r'' - m \left[ \frac{w''}{w} + (m-1) \left( \frac{w'}{w} \right)^2 + \frac{2}{\alpha + z} \frac{w'}{w} \right] r = 0 \quad (4.50)$$

These four methods of generating solutions represent only one class of solution generating methods. As mentioned previously, another form of the Hill-type equation (4.10) is a simple form of the Riccati equation (4.21). If

$$Q = u(z) + v(z) \quad (4.51)$$

then

$$Q' + Q^2 - [(u^2 + u') + 2uv + v^2 + v'] = 0 \quad (4.52)$$

The nonhomogeneous part has five terms. The third and fourth terms cancel when

$$v = -2u \quad (4.53)$$

then

$$Q^2 + Q' - [(u^2 + u') - 2u'] = 0 \quad (4.54)$$

Though seemingly trivial, this actually represents an important result.

Continuing in this fashion, choose the third and fifth term in (4.52) to cancel. This implies that

$$v = \alpha e^{-2 \int u(z) dz} \quad (4.55)$$

which corresponds to

$$Q^2 + Q' - [(u^2 + u') + \alpha^2 e^{-4 \int u(z) dz}] = 0 \quad (4.56)$$

Finally, if the fourth and fifth terms in (4.52) are chosen to add to zero, then

$$v = \frac{1}{\alpha + z} \quad (4.57)$$

and

$$Q^2 + Q' - [(u^2 + u') + \frac{2u}{\alpha + z}] = 0 \quad (4.58)$$

## THE QUADRATIC TAPER

The focus of this section is on optical fibers. They are often manufactured by applying axial tension to a heated fiber, thus stretching it. Bures et. al. [44] proposed a quadratic model to characterize such "tapers". Of course this model does not work well near the edge since the slope of the taper cannot match the model. Nonetheless, Burns et al. [45] have demonstrated experimentally that the model is valid near the center of the taper. For slowly varying tapers, the quadratic taper becomes even more reasonable. The purpose here is to solve for propagation in such tapers in terms of the powerful ABCD matrix formalism.

As previously mentioned, (4.1) must be solved. However, notice that (4.1) is similar to (4.13). Therefore, it is not difficult to see that substitution (4.14) will yield an equation not unlike (4.15). Here, we are interested in constant  $n_o$ . Therefore

$$r'' + n_2(z)/n_o r = 0 \quad (4.59)$$

However, note that for weak lenslike media, the paraxial ray equation (2.90) reduces exactly to this. Hence, the ABCD formalism is introduced by solving (4.59) and applying appropriate boundary conditions. These boundary conditions have been solved for in matrix form as the tapered complex lenslike media matrix in



table 2.1.

As shown in the introduction to this chapter,  $n_2(z)$  can be expressed in term of  $r_{cl}(z)$ . If this is done, (4.59) can be rewritten

$$r'' + 2 \frac{1 - n_{cl}/n_o}{r_{cl}^2(z)} r = 0 \quad (4.60)$$

As mentioned, the interest here lies in quadratic tapered media. Hence

$$r'' + 2 \frac{1 - n_{cl}/n_o}{(a + bz^2)^2} r = 0 \quad (4.61)$$

But the solution to this is

$$r = (a + bz^2)^{1/2} \left\{ c_1 \cos \left[ \left( \frac{ab + 2(1 - n_{cl}/n_o)}{ab} \right)^{1/2} \tan^{-1} \left( \frac{b^{1/2} z}{a^{1/2}} \right) \right] + c_2 \sin \left[ \left( \frac{ab + 2(1 - n_{cl}/n_o)}{ab} \right)^{1/2} \tan^{-1} \left( \frac{b^{1/2} z}{a^{1/2}} \right) \right] \right\} \quad (4.62)$$

This result can easily be generated by letting  $u = (a + bz^2)^{1/2}$  in (4.38), redefining the constant  $F$ , and applying (4.37) and (4.35) and (4.39). If  $a$  and  $b$  are real, and of opposite sign, the following relation may be useful

$$i \tanh^{-1} z = \tan^{-1} iz \quad (4.63)$$

It should be noted that in the limit of small  $b$ , the solution becomes the expected undamped sinusoid.

The tapered complex lenslike medium matrix in table 2.1 can be applied to (4.62). If

$$a = r_{\min} \quad (4.64)$$

$$b = (r_{\max} - r_{\min})/l^2 \quad (4.65)$$

$$\beta(z) \equiv \frac{[r_{\min}(r_{\max} - r_{\min})/l^2 + 2(1 - n_{cl}/n_o)]^{1/2}}{[r_{\min}(r_{\max} - r_{\min})/l^2]^{1/2}} \tan^{-1} \left[ \left( \frac{r_{\max}}{r_{\min}} - 1 \right)^{1/2} \frac{z}{l} \right] \quad (4.66)$$

then

$$A(z, z_1, r_{\min}, r_{\max}, l, n_{cl}, n_o) = [r_{\min} + (r_{\max} - r_{\min})(z/l)^2]^{1/2} [C_1 \cos \beta(z) + C_2 \sin \beta(z)] \quad (4.67)$$

$$B(z, z_1, r_{\min}, r_{\max}, l, n_{cl}, n_o) = [r_{\min} + (r_{\max} - r_{\min})(z/l)^2]^{1/2} [C_3 \cos\beta(z) + C_4 \sin\beta(z)] \quad (4.68)$$

$$C \equiv dA/dz \quad (4.69)$$

$$D \equiv dB/dz \quad (4.70)$$

where

$$C_1 = \frac{[r_{\min} + (r_{\max} - r_{\min})(z_1/l)^2]^{-1/2}}{[r_{\min}(r_{\max} - r_{\min})/l^2 + 2(1 - n_{cl}/n_o)]^{1/2}} \left[ \frac{(r_{\max} - r_{\min})z_1}{l^2} \sin\beta(z_1) - W \cos\beta(z_1) \right] \quad (4.71)$$

$$C_2 = - \frac{[r_{\min} + (r_{\max} - r_{\min})(z_1/l)^2]^{-1/2}}{[r_{\min}(r_{\max} - r_{\min})/l^2 + 2(1 - n_{cl}/n_o)]^{1/2}} \left[ \frac{(r_{\max} - r_{\min})z_1}{l^2} \cos\beta(z_1) + W \sin\beta(z_1) \right] \quad (4.72)$$

$$C_3 = - \frac{[r_{\min} + (r_{\max} - r_{\min})(z_1/l)^2]^{1/2}}{[r_{\min}(r_{\max} - r_{\min})/l^2 + 2(1 - n_{cl}/n_o)]^{1/2}} \sin\beta(z_1) \quad (4.73)$$

$$C_4 = \frac{[r_{\min} + (r_{\max} - r_{\min})(z_1/l)^2]^{1/2}}{[r_{\min}(r_{\max} - r_{\min})/l^2 + 2(1 - n_{cl}/n_o)]^{1/2}} \cos\beta(z_1) \quad (4.74)$$

Figure 4-1. Parameters of the Quadratic Taper

## OTHER TAPERS

Perhaps because of the difficulty in the analysis of other taper functions, the "conical" taper is most often identified as the taper function of interest in the literature. The conical taper is linear where the quadratic taper is parabolic. Propagation of light in a conical graded-index taper has been solved in terms of *ABCD*

matrices [24]. One is limited in the studies of Gaussian beam propagation by an inability to solve

$$r'' + f(z)r = 0 \quad (4.75)$$

in general. One method of obtaining solutions to this equation for various nonconstant coefficients is to convert the equation to other forms where solutions may be recognized. Another method is to invent various methods of generating solutions. Both of these methods have been given appropriate attention earlier in this chapter.

Another valuable method is to simply list out known solutions. Equation (4.59) has a multitude of applications. Some nonconstant coefficients may not be important for some applications, but are germane to others. The most important solutions for our purposes are those which are elementary (not including orthogonal polynomial) functions. A list of nonconstant coefficients that have known elementary function solutions is given in table 4.1. Similarly, those with special function solutions are given in table 4.2.

TABLE II  
ELEMENTARY SOLUTIONS OF THE HILL-TYPE EQUATION

nonconstant coefficient $f(z)$	reference
$F/(1 - \gamma z)^2$	24
$F/(1 - \gamma z/2)^4$	24
$\frac{F}{(1 + G \cos \gamma z)^4} + \frac{\gamma^2 G \cos \gamma z}{1 + G \cos \gamma z}$	39
$\frac{g_o}{1 - (z/L)^2}$	46
$\frac{g_o}{1 - (z/L)}$	46
$1 + \frac{F + G^2 - 1}{[1 + G \cos(2z)]^2}$	40
$(V^2 \operatorname{sech}^2(z/a) - B^2)/a^2$	47

TABLE III  
SPECIAL SOLUTIONS OF THE HILL-TYPE EQUATION

$f(z)$	reference
$\lambda^2 - \frac{v^2 - 1/4}{z^2}$	35, p. 362
$\frac{\lambda^2}{4z} - \frac{v^2 - 1}{4z^2}$	35, p. 362
$\lambda^2 z^{p-2}$	35, p. 362
$\lambda^2 e^{2z} - v^2$	35, p. 362
$F(1 + 2\gamma z)$	24
$-1/4 + \kappa/z + (1/4 - \mu^2)/z^2$	35, p. 505 (538)
$az^2 + bz + c$	35, p. 686
$a - 2q\cos(2z)$	35, p. 686

## SUMMARY

In the second chapter, Maxwell's equations were reduced to a wave equation. In the third chapter, the wave equation was reduced to a set of ordinary differential equations. Then this set of equations were reduced to a single equation:

$$Q_{\alpha}^2 + kn_o \frac{\partial Q_{\alpha}}{\partial z} + k^2 n_o n_{2\alpha} = 0$$

With the substitution  $Q_{\alpha} = kn_o r' / r$ , this equation becomes

$$r'' + n_{2\alpha}(z)/n_o r = 0$$

This equation has no general solution for an arbitrary nonconstant coefficient. This chapter has considered three methods to obtain solutions. First, alternate forms of the equation were realized. Second, techniques to generate solutions were considered. Finally, lists of known solutions were made. In addition to merely examining methods to solve the above equation, a physically important nonconstant coefficient is considered. The nonconstant coefficient corresponds to a quadratic taper. The above equation is solved for this taper function. The solutions are elementary.

## **CHAPTER V**

### **MODELING AND APPLICATIONS OF TAPERED WAVEGUIDES**

#### **INTRODUCTION**

In chapter II it was argued that the analysis there agrees well with experiment. It is the purpose of this chapter to demonstrate the usefulness of such analyses. Applications of tapered waveguides include uses such as power concentration, image minification, coupling and mode conversion [48]. Tapered waveguides have also proven as useful models for the tapered cone-photoreceptor of the human eye [49]. The analyses here are general enough to include laser amplifiers and oscillators. Therefore, there exist other applications including mode control and tapered gain guiding. For other applications see Ref. 24 and references therein.

#### **SYMMETRIC QUADRATIC TAPER**

As previously mentioned, fiber tapers are often constructed by applying an axial tension to a heated fiber. The resultant shape is a symmetric function with an absolute minimum at the center of the taper. Bures et. al. [44] proposed an exponential to model this function. Burns [45] suggested a quadratic function. He demonstrated the validity of his model experimentally. The construction of quadratic-index waveguides is a delicate process. Sometimes defects occur which may affect the longitudinal index or loss distribution. Such defects can be modeled to first order as quadratic variations. Hence, it would be prudent to solve

for the propagation of Gaussian beams in this type of taper. Such is the purpose here.

The propagation equation is

$$r'' + 2 \frac{1 - n_{cl}/n_o}{(a + bz^2)^2} r = 0 \quad (5.1)$$

But these solutions were found in chapter IV. Hence, the symmetric quadratic taper can be viewed as a special case of the generalized quadratic taper. If  $A$ ,  $B$ ,  $C$ , and  $D$  represent the elements found there, then under the constraints that

$$\Gamma_o = \Gamma(-l < z < l, z_1 = -l, r_{\min} = R_{\min}, r_{\max} = R_{\max}, l, n_{cl}, n_o) \quad (5.2)$$

the  $ABCD$  matrix for the symmetric taper is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_o & B_o \\ C_o & D_o \end{bmatrix} \quad (5.3)$$

where  $\Gamma$  represents either  $A$ ,  $B$ ,  $C$ , or  $D$ . A typical taper in the middle of an optical waveguide is shown in figure 5.1.

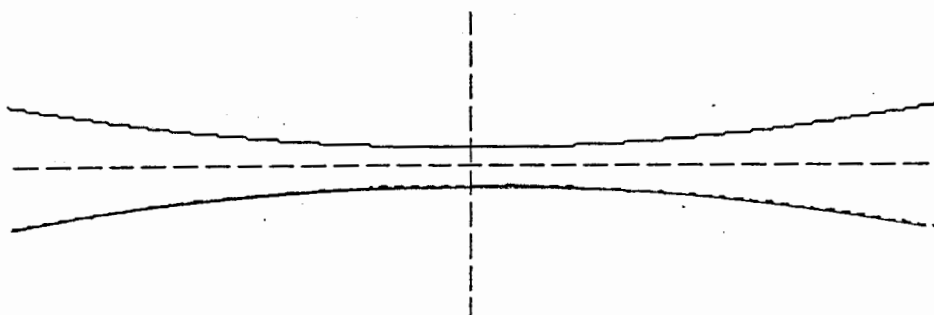


Figure 5-1. Symmetric Quadratic Taper.

Beam amplitude and phase displacement can be calculated from chapter II. Similarly, spotsize and phase front curvature can be calculated. Higher-order mode formulas are given in chapter III. Recall that ray displacement for real lenslike



media is governed by

$$r = Ar_1 + Br'_1 \quad (5.4)$$

As a representative example, such ray displacement is sketched in figure 5.2.

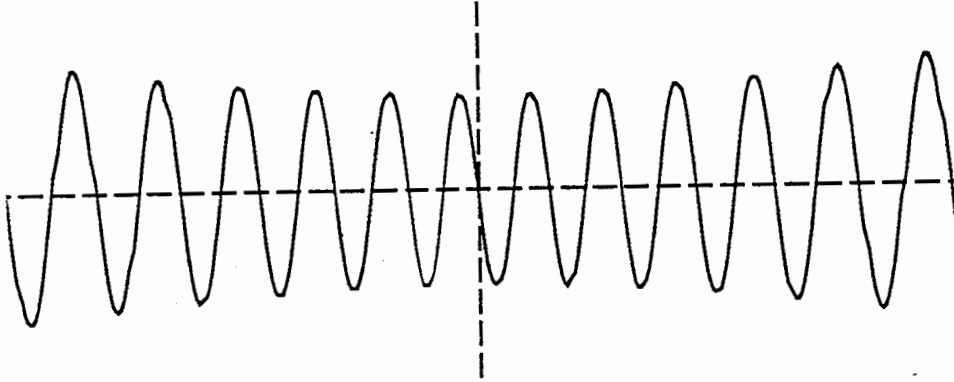


Figure 5-2. Ray Displacement in a Lenslike Symmetric Quadratic Taper.

### SPLINE COUPLER

It is clear that if  $z$  was only allowed to vary from  $-l$  to  $0$ , the symmetric quadratic taper would reduce to a simple coupler. This coupler has zero slope at the "far end" of the taper, but not at the "near end". The purpose of this section is to consider a coupler that has zero slope at both ends.

Curves can be fit to a given finite number of discrete data points. It is not always clear what function best fits the data. Lagrange introduced a generalized method to fit data using polynomials [50]. Often it is the case that a large number of data points is given, therefore it requires a large order polynomial to obtain a reasonable result. With this in mind, one generally resorts to the computational power of a digital computer.

There is another popular method. Instead of approximating the whole curve by a polynomial, one can approximate the curves between each of the data points as polynomials. Most often third degree polynomials are used, and such a method is therefore called "cubic spline interpolation" [50]. Sometimes linear splines [51] and less often constant splines [52] are also used.

The purpose here is to introduce the use of quadratic splines for analytical calculations. The difficulty with the constant spline is that one cannot obtain continuity. Similarly, the difficulty with the linear spline is that one cannot obtain continuity of the first derivative. It has already been suggested that a quadratic function is a realistic model of a fiber taper. Therefore, a quadratic spline should be an even more realistic model. Such splines are well suited to the *ABCD* formalism since one merely need multiply matrices together to obtain beam propagation characteristics. Similar arguments have suggested that such "composite elements" are useful for the propagation of light through periodically tapered media [23].

Figure 5.3 demonstrates how a coupler can be obtained by quadratic spline with three data points. Note that each of the quadratics have their maxima/minima at the origin. This is necessary because of the functional form of the quadratic:

$$r_{cl,\alpha} = r_{min,\alpha} + (r_{max,\alpha} - r_{min,\alpha})(z/l)^2 \quad (5.5)$$

where  $\alpha$  is either 1 or 2.

Again, it is desired to match position and slope. Therefore, the boundary conditions are

$$r_{cl,1}(l_1) = r_{cl,2}(-l_2) \quad (5.6)$$

$$r'_{cl,1}(l_1) = r'_{cl,2}(-l_2) \quad (5.7)$$

If  $R_{max}$  is the initial fiber radius,  $R_{min}$  is the final fiber radius,  $L$  is the total length of the taper, and  $l_1$  ( $0 < l_1 < L$ ) is the longitudinal distance which may be considered as an adjustable parameter, then

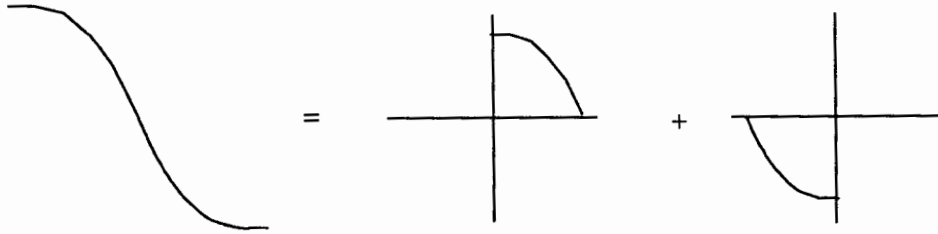


Figure 5-3. Spline Coupler

$$R_{\max} \equiv r_{\min, 1} \quad (5.8)$$

$$R_{\min} \equiv r_{\min, 2} \quad (5.9)$$

$$L \equiv l_1 + l_2 \quad (5.10)$$

$$\Gamma_1 = \Gamma(z_1 = 0, 0 < z < l_1, r_{\min} = R_{\max}, r_{\max} = (1 - l_1/L)R_{\max} + R_{\min}l_1/L, \quad (5.11)$$

$$l = l_1, n_{cl}, n_o)$$

$$\Gamma_2 = \Gamma(z_1 = -(L - l_1), -(L - l_1) < z < 0, r_{\min} = R_{\min}, \quad (5.12)$$

$$r_{\max} = (1 - l_1/L)R_{\max} + R_{\min}l_1/L, l = L - l_1, n_{cl}, n_o)$$

The final  $ABCD$  matrix can now be calculated. If

$$0 < z < l_1 \quad (5.13)$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1(z) & B_1(z) \\ C_1(z) & D_1(z) \end{bmatrix} \quad (5.14)$$

$$l_1 < z < L \quad (5.15)$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_2(z-2l_1) & B_2(z-2l_1) \\ C_2(z-2l_1) & D_2(z-2l_1) \end{bmatrix} \begin{bmatrix} A_1(l_1) & B_1(l_1) \\ C_1(l_1) & D_1(l_1) \end{bmatrix} \quad (5.16)$$

## SYMMETRIC SPLINE TAPER

One difficulty with the symmetric quadratic taper is that the ends of the taper do not have zero slope. By using the splining techniques developed in the previous sections, this difficulty can be rectified. For simplicity, boundary conditions are matched halfway between the center and the end of the taper:

$$r_{cl, 1}(l) = r_{cl, 2}(-l) \quad (5.17)$$

$$r'_{cl,1}(l) = r'_{cl,2}(-l) \quad (5.18)$$

$$r_{cl,2}(l) = r_{cl,3}(-l) \quad (5.19)$$

$$r'_{cl,2}(l) = r'_{cl,3}(-l) \quad (5.20)$$

These boundary conditions can be used to calculate  $r_{min,1,2,3}$  and  $r_{max,1,2,3}$ . The appropriateness of the boundary conditions can be seen from figure 5.4.

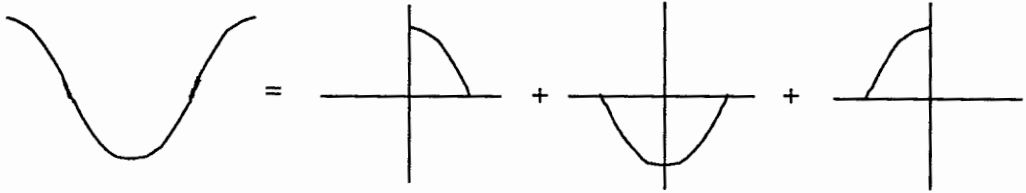


Figure 5-4. Spline Taper

It should also be clear from the figure that  $r_{cl,1}(0) = r_{cl,3}(0) = R_{max}$  and  $r_{cl,2}(0) = R_{min}$ .

These boundary conditions and definitions imply that

$$\Gamma_1 = \Gamma(z_1 = 0, 0 < z < l_1, r_{min} = R_{max}, r_{max} = (R_{max} + R_{min})/2, \quad (5.21)$$

$$l = L/4, n_{cl}, n_o)$$

$$\Gamma_2 = \Gamma(z_1 = -L/4, -L/4 < z < L/4, r_{min} = R_{min}, r_{max} = (R_{max} + R_{min})/2, \quad (5.22)$$

$$l = L/4, n_{cl}, n_o)$$

$$\Gamma_3 = \Gamma(z_1 = -L/4, -L/4 < z < 0, r_{min} = R_{max}, r_{max} = (R_{max} + R_{min})/2, \quad (5.23)$$

$$l = L/4, n_{cl}, n_o)$$

Finally, the  $ABCD$  matrices can be calculated:

$$0 < z < L/4 \quad (5.24)$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1(z) & B_1(z) \\ C_1(z) & D_1(z) \end{bmatrix} \quad (5.25)$$

$$L/4 < z < 3L/4 \quad (5.26)$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_2(z-2l) & B_2(z-2l) \\ C_2(z-2l) & D_2(z-2l) \end{bmatrix} \begin{bmatrix} A_1(l) & B_1(l) \\ C_1(l) & D_1(l) \end{bmatrix} \quad (5.27)$$

$$3L/4 < z < L \quad (5.28)$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_3(z-4l) & B_3(z-4l) \\ C_3(z-4l) & D_3(z-4l) \end{bmatrix} \begin{bmatrix} A_2(l) & B_2(l) \\ C_2(l) & D_2(l) \end{bmatrix} \begin{bmatrix} A_1(l) & B_1(l) \\ C_1(l) & D_1(l) \end{bmatrix} \quad (5.29)$$

## PERIODIC TAPERS AND TAPERED RESONATORS

From figure 5.5 it is easy to see that two splined parabolas reasonably approximate a single period of a sinusoid. This has been argued more quantitatively in Ref. 35.

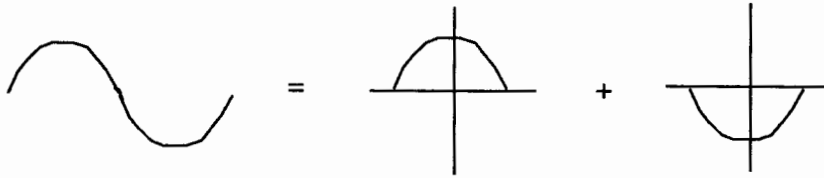


Figure 5-5. Spline Periodic Waveguide

Continuing as in the previous sections, boundary conditions must be stated and matched:

$$r_{cl,1}(l) = r_{cl,2}(-l) \quad (5.30)$$

$$r'_{cl,1}(l) = r'_{cl,2}(-l) \quad (5.31)$$

The boundary conditions combined with the definitions

$$r_{cl,1}(0) = R_{\max} \quad (5.32)$$

$$r_{cl,2}(0) = R_{\min} \quad (5.33)$$

imply that

$$\Gamma_1 = \Gamma(z_1 = -T/4, -T/4 < z < T/4, r_{\min} = R_{\max}, r_{\max} = (R_{\max} + R_{\min})/2, l = T/4, n_{cl}, n_o) \quad (5.34)$$

$$\Gamma_2 = \Gamma(z_1 = -T/4, -T/4 < z < T/4, r_{\min} = R_{\min}, r_{\max} = (R_{\max} + R_{\min})/2, l = T/4, n_{cl}, n_o) \quad (5.35)$$

The  $ABCD$  matrices for a single period are

$$0 < z < T/2 \quad (5.36)$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1(z-T/4) & B_1(z-T/4) \\ C_1(z-T/4) & D_1(z-T/4) \end{bmatrix} \quad (5.37)$$

$$T/2 < z < T \quad (5.38)$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_2(z-3T/4) & B_2(z-3T/4) \\ C_2(z-3T/4) & D_2(z-3T/4) \end{bmatrix} \begin{bmatrix} A_1(T/4) & B_1(T/4) \\ C_1(T/4) & D_1(T/4) \end{bmatrix} \quad (5.39)$$

To determine the beam propagation characteristics after completing  $n$  periods, and then part or all the way through another, simply postmultiply the above matrices by

$$\left[ \begin{bmatrix} A_2(z-3T/4) & B_2(z-3T/4) \\ C_2(z-3T/4) & D_2(z-3T/4) \end{bmatrix} \begin{bmatrix} A_1(T/4) & B_1(T/4) \\ C_1(T/4) & D_1(T/4) \end{bmatrix} \right]^n \quad (5.40)$$

Sylvester's theorem mentioned in the second chapter governed the  $n$ th power of a  $2 \times 2$  matrix:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^n = \frac{1}{\sin \theta} \begin{bmatrix} A \sin(n\theta) - \sin[(n-1)\theta] & B \sin(n\theta) \\ C \sin(n\theta) & D \sin(n\theta) - \sin[(n-1)\theta] \end{bmatrix} \quad (5.41)$$

where  $\cos \theta = \frac{1}{2}(A + D)$ . It should be stressed that

$$\{[A][B]\}^n \neq [A]^n[B]^n \quad (5.42)$$

for matrices.

## SUMMARY

In the forth chapter, a physically interesting and important taper function was identified, and solutions to the electric field were obtained. This taper function was the quadratic taper, and it was examined at the beginning of this chapter. A coupler can be obtained by simply using only half of the taper. The difficulty is that only one end has zero slope. However, two parabolas (quadratics) can be concatenated, or splined together as demonstrated in figure 5-3. This method of splining can be used to obtain a more realistic taper function. Using these techniques, a periodic waveguide can also be simulated.

## CHAPTER VI

### CONCLUSION

For the first time, exact solutions to the paraxial wave equation have been found for off-axis multimode light beam propagation in lenslike media with transverse spatial gain or loss variation. The analysis is general enough to include elliptical input beams in tapered media. The electric and magnetic fields can be solved in terms of a beam parameter which is governed by a Riccati equation. The Riccati equation cannot be solved in general for an arbitrary taper function. To alleviate this difficulty, a physically important taper function has been identified, and the corresponding Riccati equation has been solved. Also, methods of obtaining solutions to Riccati equations in general have been examined in detail. Finally, composite tapers are considered. These composite, or spline tapers can be used to simulate nearly any type of taper encountered in practice.

Future work may be concentrated in a variety of areas. More work could be done on the interpretation of the  $a$  and  $b$  parameters governing multimode propagation. The ramifications of the  $a$  and  $b$  transformations may also be considered. As an example, it is easy to obtain stability criteria for the two transformations. New and interesting solutions to the Riccati equation can be found via methods in chapter IV. One may apply the boundary conditions implied by a cladding at finite radius to obtain more mode information. There were a variety of approximations and constraints made leading up to the paraxial wave equation.

A relaxing of these approximations and constraints while keeping the generality of arbitrarily tapered media is also work for the future.



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